PERIODIC ORBITS OF HAMILTONIAN SYSTEMS:
APPLICATIONS TO PERTURBED KEPLER PROBLEMS

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ABSTRACT. We provide for a class of Hamiltonians in the action–angle variables sufficient conditions for studying their periodic orbits. We improve this result for the general perturbed spatial Keplerian Hamiltonians with axial symmetry. Finally, we apply these results for studying the periodic orbits of the Matese–Whitman Hamiltonian, of the spatial anisotropic Hamiltonian and of the spatial generalized van der Waals Hamiltonian.

1. Introduction and statement of the main results

We consider the following class of Hamiltonians in the action–angle variables

\begin{equation}
\mathcal{H}(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n) = \mathcal{H}_0(I_1) + \varepsilon \mathcal{H}_1(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n),
\end{equation}

where $\varepsilon$ is a small parameter.

As usual the Poisson bracket of the functions $f(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)$ and $g(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)$ is

$$\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial I_i} - \frac{\partial f}{\partial I_i} \frac{\partial g}{\partial \theta_i} \right).$$

The next result provides sufficient conditions for computing periodic orbits of the Hamiltonian system associated to the Hamiltonian (1).

**Theorem 1.** We define

$$\langle \mathcal{H}_1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_1(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n) d\theta_1,$$

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and we consider the differential system

\[
\begin{align*}
\frac{dI_i}{d\theta_1} &= \varepsilon \{I_i, \langle \mathcal{H}_1 \rangle \} = \varepsilon f_{i-1}(I_2, \ldots, I_n, \theta_2, \ldots, \theta_n) \quad i = 2, \ldots, n, \\
\frac{d\theta_i}{d\theta_1} &= \varepsilon \{\theta_i, \langle \mathcal{H}_1 \rangle \} = \varepsilon f_{i+n-2}(I_2, \ldots, I_n, \theta_2, \ldots, \theta_n) \quad i = 2, \ldots, n.
\end{align*}
\]  

(2)

This system is a Hamiltonian system with Hamiltonian \(\varepsilon \langle \mathcal{H}_1 \rangle\). Fixed the energy level \(\mathcal{H} = h^* < 0\), if \(\varepsilon \neq 0\) is sufficiently small then for every equilibrium point \(p = (I^0_2, \ldots, I^0_n, \theta^0_2, \ldots, \theta^0_n)\) of system (2) satisfying that

\[
\det \left( \frac{\partial (f_1, \ldots, f_{2n-2})}{\partial (I_2, \ldots, I_n, \theta_2, \ldots, \theta_n)} \right)_{(I_2, \ldots, I_n, \theta_2, \ldots, \theta_n) = (I^0_2, \ldots, I^0_n, \theta^0_2, \ldots, \theta^0_n)} \neq 0,
\]

there exists a \(2\pi\)-periodic solution \(\gamma_\varepsilon(\theta_1) = (I_1(\theta_1, \varepsilon), \ldots, I_n(\theta_1, \varepsilon), \theta_2(\theta_1, \varepsilon), \ldots, \theta_n(\theta_1, \varepsilon))\) of the Hamiltonian system associated to the Hamiltonian (1) taking as independent variable the angle \(\theta_1\) such that \(\gamma_\varepsilon(0) \to (\mathcal{H}_0^{-1}(h^*), I^0_2, \ldots, I^0_n, \theta^0_2, \ldots, \theta^0_n)\) when \(\varepsilon \to 0\). The stability or instability of the periodic solution \(\gamma_\varepsilon(\theta_1)\) is given by the stability or instability of the equilibrium point \(p\) of system (2). In fact, the equilibrium point \(p\) has the stability behavior of the Poincaré map associated to the periodic solution \(\gamma_\varepsilon(\theta_1)\).

Theorem 1 will be proved in section 2.

The next objective of the present work is to study the periodic orbits of the Hamiltonian system with the perturbed Keplerian Hamiltonian of the form

\[
\mathcal{H} = \frac{1}{2} \left( P_1^2 + P_2^2 + P_3^2 \right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} + \varepsilon P_1 (Q_1^2 + Q_2^2, Q_3).
\]

(3)

Note that the perturbation is symmetric with respect to the \(Q_3\)-axis. It is easy to check that the third component \(K = Q_1 P_2 - Q_2 P_1\) of the angular momentum is a first integral of the Hamiltonian system associated to the Hamiltonian (3). We use this second first integral to simplify the analysis of the given axially symmetric Keplerian perturbed system.

In the following we use the Delaunay variables for studying easily the periodic orbits of the Hamiltonian system associated to the Hamiltonian (3), see \([4, 13]\) for more details on the Delaunay variables. Thus, in Delaunay variables the Hamiltonian (3) has the form

\[
\mathcal{H} = -\frac{1}{2L^2} + \varepsilon \mathcal{P}(l, g, k, L, G, K) = -\frac{1}{2L^2} + \varepsilon \mathcal{P}(l, g, L, G, K),
\]

(4)

where \(l\) is the mean anomaly, \(g\) is the argument of the perigee of the unperturbed elliptic orbit measured in the invariant plane, \(k\) is the longitude of the node, \(L\) is the square root of the semi-major axis of the unperturbed elliptic orbit, \(G\) is the modulus of the total angular momentum and \(K\) is the third component of the angular momentum. Moreover, \(\mathcal{P}\) is the perturbation...
obtained from the perturbation $P_1$ using the transformation to Delaunay variables, namely
\begin{align}
Q_1 &= r \left( \cos(f + g) \cos k - c \sin(f + g) \sin k \right), \\
Q_2 &= r \left( \cos(f + g) \sin k + c \sin(f + g) \cos k \right), \\
Q_3 &= rs \sin(f + g),
\end{align}
with
\begin{align}
c &= \frac{K}{G}, \quad s^2 &= 1 - \frac{K^2}{G^2}.
\end{align}

The true anomaly $f$ and the eccentric anomaly $E$ are auxiliary quantities defined by the relations
\begin{align}
\sqrt{1 - e^2} &= \frac{G}{L}, \quad r = a(1 - e \cos E), \quad l = E - e \sin E.
\end{align}

\begin{align}
\sin f &= \frac{a \sqrt{1 - e^2} \sin E}{r}, \quad \cos f &= \frac{a(\cos E - e)}{r},
\end{align}
where $e$ is the eccentricity of the unperturbed elliptic orbit.

Note that the angular variable $k$ is a cyclic variable for the Hamiltonian (4), and consequently $K$ is a first integral of the Hamiltonian system as we already knew.

The family of Hamiltonians (4) is a particular subclass of the Hamiltonians (1) with $H_1 = P$. We denote by $\langle P \rangle$ the averaged map of $P$ with respect to the mean anomaly $l$, i.e.,
\begin{align}
\langle P \rangle = \frac{1}{2\pi} \int_0^{2\pi} P(l, g, L, G, K) dl = \frac{1}{2\pi} \int_0^{2\pi} P(E, g, L, G, K)(1 - e \cos E) dE.
\end{align}

We remark that the map $\langle P \rangle$ only depends on the angle $g$ and the three action variables $L, G, K$. By Theorem 1 at the energy level $H = h^* < 0$ with angular momentum $K = k^*$ the differential equations (2) with respect to the mean anomaly $l$ are
\begin{align}
\frac{dG}{dl} &= \varepsilon \left\{ \frac{G}{H_0(H_0^{-1}(h^*))} \right\} = -\varepsilon(-2h^*)^{3/2} \frac{\partial (\langle P \rangle)}{\partial g} = -\varepsilon f_1(g, G, K), \\
\frac{dg}{dl} &= \varepsilon \left\{ \frac{g}{H_0(H_0^{-1}(h^*))} \right\} = \varepsilon(-2h^*)^{3/2} \frac{\partial (\langle P \rangle)}{\partial G} = \varepsilon f_2(g, G, K), \\
\frac{dk}{dl} &= \varepsilon \left\{ \frac{k}{H_0(H_0^{-1}(h^*))} \right\} = \varepsilon(-2h^*)^{3/2} \frac{\partial (\langle P \rangle)}{\partial K} = \varepsilon f_3(g, G, K).
\end{align}

Note that we do not write the differential equation $dK/dt = 0$.

Now we are ready to state a corollary of Theorem 1 which provides sufficient conditions for the existence and the kind of stability of the periodic orbits in the perturbed Kepler problems with axial symmetry.
Corollary 2. System (6) is the Hamiltonian system taking as independent variable the mean anomaly \( l \) of the Hamiltonian (3) written in Delaunay variables on the fixed energy level \( \mathcal{H} = h^* < 0 \) and on the fixed third component of the angular momentum \( K = k^* \). If \( \varepsilon \neq 0 \) is sufficiently small then for every zero \( p = (g_0, G_0, k^*) \) of the system \( f_i(g, G, K) = 0 \) for \( i = 1, 2, 3 \) satisfying that
\[
\det \left( \frac{\partial (f_1, f_2, f_3)}{\partial (g, G, K)} \right)_{(g, G, K) = (g_0, G_0, k^*)} \neq 0,
\]
there exists a \( 2\pi \)-periodic solution \( \gamma_\varepsilon(l) = (g(l, \varepsilon), k(l, \varepsilon), L(l, \varepsilon), G(l, \varepsilon), K(l, \varepsilon)) = k^* \) such that \( \gamma_\varepsilon(0) \to (g_0, *, \sqrt{-2h^*}, G_0, k^*) \) when \( \varepsilon \to 0 \), where the * in the second component denotes an unknown value. The stability or instability of the periodic solution \( \gamma_\varepsilon(l) \) is given by the stability or instability of the equilibrium point \( p \) of system (6). In fact, the equilibrium point \( p \) has the stability behavior of the Poincaré map associated to the periodic solution \( \gamma_\varepsilon(l) \).

There are many articles studying different perturbed Keplerian problems, see for instance [8, 9, 15] and the papers quoted therein.

In what follows we shall study three Hamiltonian systems with Hamiltonian of the form (4). The first one will be the Matese–Whitman Hamiltonian system modeling the galactic tidal interaction with the Oort comet cloud. The second one is the spatial anisotropic Kepler problem which analyzes the isotropy of the space. Finally, we consider the spatial generalized van der Waals Hamiltonian system modeling the dynamical symmetries of the perturbed hydrogen atom.

The Hamiltonian system associated to the Hamiltonian
\[
\mathcal{H} = \frac{1}{2} (P_1^2 + P_2^2 + P_3^2) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} + \varepsilon Q_3^2
\]
was proposed in the seminal papers of Matese and Whitman [11, 12] for studying the dynamics of the Oort cloud [14], see also [10]. In fact it is a simple dynamical system which consider the heliocentric Kepler problem perturbed by the quadratic tidal potential of the Galaxy. Note that the Hamiltonian (8) is a particular case of the Hamiltonian (3).

We have the following result on the periodic orbits and their kind of stability for the Hamiltonian system associated to the Hamiltonian (8).

**Theorem 3.** On every energy level \( \mathcal{H} = h^* < 0 \) and for the third component of the angular momentum \( K = k^* = 0 \), the Matese–Whitman Hamiltonian system associated to the Hamiltonian (8) for \( \varepsilon \neq 0 \) sufficiently small has one \( 2\pi \)-periodic solution \( \gamma_\varepsilon(l) = (g(l, \varepsilon), k(l, \varepsilon)), L(l, \varepsilon), G(l, \varepsilon), K(l, \varepsilon)) \) such that
\[
\gamma_\varepsilon(0) \to \left( \frac{1}{2} \arccos \left( \frac{3}{5} \right), *, \frac{1}{\sqrt{-2h^*}}, \frac{1}{\sqrt{-2h^*}}, 0 \right) \text{ when } \varepsilon \to 0.
\]
This periodic orbit has a stable manifold of dimension 2 and an unstable of dimension 1.
Theorem 3 is proved in section 3.

We note that the values \( \pm 26.6... = \pm \arccos(3/5)/2 \) also appear in the paper [11], being there a boundary for the argument of the perigee of the periodic orbits of the Hamiltonian system with Hamiltonian (8).

Now we shall analyze the so called anisotropic spatial Kepler problem given by the Hamiltonian

\[
H = \frac{1}{2} \left( P_1^2 + P_2^2 + P_3^2 \right) - \frac{1}{\sqrt{\mu(Q_1^2 + Q_2^2 + Q_3^2)}}
\]

where \( \mu \) is a positive real parameter. This problem originally comes from the quantum mechanic, see for instance [7, 2, 3].

We want to study its periodic orbits for values of the parameter \( \mu \) close to 1. Therefore define \( \varepsilon = 1 - \mu \), expanding in Taylor series in \( \varepsilon \) we obtain

\[
H = \frac{1}{2} \left( P_1^2 + P_2^2 + P_3^2 \right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} \quad \varepsilon \frac{Q_1^2 + Q_2^2}{2(Q_1^2 + Q_2^2 + Q_3^2)^{3/2}} + O(\varepsilon^2).
\]

Note that this Hamiltonian is of the form (3).

**Theorem 4.** On every energy level \( H = h^* < 0 \) and for the third component of the angular momentum \( K = k^* = 0 \), the spatial anisotropic Kepler Hamiltonian system associated to the Hamiltonian (9) for \( \varepsilon \neq 0 \) sufficiently small has one \( 2\pi \)-periodic solution \( \gamma_\varepsilon(l) = (g(l, \varepsilon), k(l, \varepsilon), L(l, \varepsilon), G(l, \varepsilon), K(l, \varepsilon)) \) such that

\[
\gamma_\varepsilon(0) \rightarrow \left( \frac{\pi}{4}, *, \frac{1}{\sqrt{-2h^*}}, \frac{1}{\sqrt{-2h^*}}, 0 \right) \quad \text{when} \quad \varepsilon \rightarrow 0.
\]

This periodic orbit has a stable manifold of dimension 2 and an unstable of dimension 1.

Theorem 4 is proved in section 4.

The generalized van der Waals Hamiltonian system was proposed in the paper [1] via the following Hamiltonian with \( \beta \in \mathbb{R} \)

\[
H = \frac{1}{2} \left( P_1^2 + P_2^2 + P_3^2 \right) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} + \varepsilon \left( Q_1^2 + Q_2^2 + \beta^2 Q_3^2 \right).
\]

Note that this Hamiltonian is of the form (3). For more references on this Hamiltonian system see the ones quoted in [6].

**Theorem 5.** On every energy level \( H = h^* < 0 \) and for the third component of the angular momentum \( K = k^* = 0 \), the spatial van der Waals Hamiltonian system associated to the Hamiltonian (10) for \( \varepsilon \neq 0 \) sufficiently small has:

(a) One \( 2\pi \)-periodic solution \( \gamma_\varepsilon(l) = (g(l, \varepsilon), k(l, \varepsilon), L(l, \varepsilon), G(l, \varepsilon), K(l, \varepsilon)) \) such that

\[
\gamma_\varepsilon(0) \rightarrow \left( \frac{1}{2} \arccos \left( \frac{3(\beta^2 + 1)}{5(\beta^2 - 1)} \right), *, \frac{1}{\sqrt{-2h^*}}, \frac{1}{\sqrt{-2h^*}}, 0 \right) \quad \text{when} \quad \varepsilon \rightarrow 0,
\]

if \( \beta \in (-\infty, -2) \cup (-1/2, 1/2) \cup (2, \infty) \). This periodic orbit has a stable manifold of dimension 2 and an unstable of dimension 1 if \( \beta \in \)
(-1/2, 1/2), and has a stable manifold of dimension 1 and an unstable manifold of dimension 2 if $\beta \in (-\infty, -2) \cup (2, \infty)$.

(b) Four $2\pi$-periodic solutions $\gamma_\varepsilon(l) = (g(l, \varepsilon), k(l, \varepsilon), L(l, \varepsilon), G(l, \varepsilon), K(l, \varepsilon))$ such that

$$
\gamma_\varepsilon(0) \rightarrow \left( \pm \frac{\pi}{2}, *, \frac{1}{\sqrt{-2h^*}}, \frac{1}{2} \sqrt{\frac{5}{-2h^*}}, \pm \frac{1}{4} \sqrt{\frac{5(1-4\beta^2)}{-2h^*(1-\beta^2)}} \right) \text{ when } \varepsilon \rightarrow 0,
$$

if $\beta \in (-1, -1/2) \cup (1/2, 1)$.

Theorem 5 is proved in section 5.

The result of statement (a) of Theorem 5 was already obtained using cylindrical coordinates in [6].

The stability or instability of the four periodic orbits of statement (b) of Theorem 5 can be determined analyzing the eigenvalues of the corresponding Jacobian matrices, but since the expression of these eigenvalues are huge and depend on the two parameters $h^*$ and $\beta$, this study is a long task that we do not do here.

We remark that when $(\beta^2 - 1)(\beta^2 - 4)(\beta^2 - 1/4) = 0$ we cannot apply the averaging theory for finding periodic orbits because for these values of $\beta$ the van der Waals Hamiltonian system is integrable and it has a continuum of periodic orbits and consequently the Jacobian (17) is zero, see for the integrability of the van der Waals Hamiltonian system [5]. Therefore, the averaging method when cannot be applied for finding periodic orbits provides a suspicion that for such values of the parameter the system could be integrable.

**Remark 6.** The periodic orbit which persists for $\varepsilon \neq 0$ sufficiently small in Theorems 3, 4 and 5(a) bifurcates from a circular periodic orbit of the unperturbed Kepler problem contained in a plane containing the $Q_3$–axis. The four periodic orbits of Theorem 5(b) bifurcate from elliptic period orbits of the unperturbed Kepler problem.

### 2. AVERAGED HAMILTONIAN IN ACTION–ANGLE VARIABLES

The Hamiltonian system associated to the Hamiltonian (1) can be written as

$$
\frac{dI_i}{dt} = \varepsilon \{I_i, \mathcal{H}_1\} = -\varepsilon \frac{\partial \mathcal{H}_1}{\partial \theta_i}, \quad i = 1, ..., n,
$$

$$
\frac{d\theta_i}{dt} = \varepsilon \{\theta_i, \mathcal{H}_1\} = \varepsilon \frac{\partial \mathcal{H}_1}{\partial I_i}, \quad i = 2, ..., n,
$$

$$
\frac{d\theta_1}{dt} = \mathcal{H}_0'(I_1) + \varepsilon \{\theta_1, \mathcal{H}_1\} = \mathcal{H}_0'(I_1) + \varepsilon \frac{\partial \mathcal{H}_1}{\partial I_1}.
$$
Lemma 7. Taking as new independent variable the variable $\theta_1$ we have in the fixed energy level $H = h^* < 0$ that the differential system (11) becomes

\[
\begin{align*}
\frac{dI_i}{d\theta_1} &= \varepsilon \frac{\{I_i, H_1\}}{H_0'(H_0^{-1}(h^*))} + O(\varepsilon^2), \quad i = 2, \ldots, n \\
\frac{d\theta_i}{d\theta_1} &= \varepsilon \frac{\{\theta_i, H_1\}}{H_0'(H_0^{-1}(h^*))} + O(\varepsilon^2), \quad i = 2, \ldots, n
\end{align*}
\]

with $I_1 = H_0^{-1}(h^*) + O(\varepsilon)$ if $H_0'(H_0^{-1}(h^*)) \neq 0$.

Proof. Taking as new independent variable $\theta_1$, the equations (11) become

\[
\begin{align*}
\frac{dI_i}{d\theta_1} &= \frac{\varepsilon \{I_i, H_1\}}{H_0'(I_1) + \varepsilon \{\theta_1, H_1\}} = \frac{\varepsilon \{I_i, H_1\}}{H_0'(I_1)} + O(\varepsilon^2) \quad i = 1, \ldots, n, \\
\frac{d\theta_i}{d\theta_1} &= \frac{\varepsilon \{\theta_i, H_1\}}{H_0'(I_1) + \varepsilon \{\theta_1, H_1\}} = \frac{\varepsilon \{\theta_i, H_1\}}{H_0'(I_1)} + O(\varepsilon^2) \quad i = 2, \ldots, n.
\end{align*}
\]

Fixing the energy level of $H = h^* < 0$ we obtain $h^* = H_0(I_1) + \varepsilon H_1(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)$. Using the Implicit Function Theorem and the fact that $H_0'(H_0^{-1}(h^*)) \neq 0$, for $\varepsilon$ sufficiently small, we get $I_1 = H_0^{-1}(h^*) + O(\varepsilon)$, and the equations are reduced to (12).

Proof of Theorem 1. The averaged system in the angle $\theta_1$ obtained from (12) is

\[
\begin{align*}
\frac{dI_i}{d\theta_1} &= \frac{1}{2\pi} \frac{\varepsilon}{H_0'(H_0^{-1}(h^*))} \int_0^{2\pi} \frac{\partial H_1}{\partial \theta_i} d\theta_1 \quad i = 2, \ldots, n, \\
\frac{d\theta_i}{d\theta_1} &= \frac{1}{2\pi} \frac{\varepsilon \{\theta_i, H_1\}}{H_0'(H_0^{-1}(h^*))} \int_0^{2\pi} \frac{\partial H_1}{\partial I_i} d\theta_1 \quad i = 2, \ldots, n.
\end{align*}
\]

See the Appendix for a short introduction to the averaging theory used in this paper.

Since,

\[
\begin{align*}
\frac{\partial \langle H_1 \rangle}{\partial \theta_i} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial H_1}{\partial \theta_i} d\theta_1 \quad i = 2, \ldots, n, \\
\frac{\partial \langle H_1 \rangle}{\partial I_i} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial H_1}{\partial I_i} d\theta_1 \quad i = 2, \ldots, n,
\end{align*}
\]
the differential system (13) becomes
\[
\frac{dI_i}{d\theta_1} = -\varepsilon \frac{\partial \langle H_1 \rangle}{\partial \theta_i} = \varepsilon \frac{\{I_i, \langle H_1 \rangle\}}{H'_0(h_0^{-1}(h))} \quad i = 2, \ldots, n, \\
\frac{d\theta_i}{d\theta_1} = \frac{\varepsilon}{H'_0(h_0^{-1}(h))} \frac{\partial \langle H_1 \rangle}{\partial I_i} = \varepsilon \frac{\{\theta_i, \langle H_1 \rangle\}}{H'_0(h_0^{-1}(h))} \quad i = 2, \ldots, n,
\]
which coincides with the system (2).

The rest of the proof of the theorem follows easily from the averaged system (2) and Theorem 8 of the Appendix. \(\square\)

3. The Matese–Whitman problem

Now the function \(P(E, g, L, G, K)\) given in (4) is
\[
\frac{(G^2 - K^2)(e \cos E - 1)^2 L^4}{2G^2} - \frac{L^4(G^2 - K^2)(e - \cos E)^2 \cos^2 g}{2G^2} \\
+ \frac{2L^3(G^2 - K^2)(e - \cos E) \cos g \sin E \sin g}{G}
\]
\[
+ \frac{L^2(G^2 - K^2) \cos^2 g \sin^2 E - \frac{1}{2} L^2(G^2 - K^2) \sin^2 E \sin^2 g}{G}
\]
Its averaged function \(\langle P \rangle\) with respect the mean anomaly \(l\) is
\[
\langle P \rangle = \frac{1}{2\pi} \int_0^{2\pi} P(E, g, L, G, K)(1 - e \cos E) dE
\]
\[
= \frac{L^2(G^2 - K^2)(5L^2 - 3G^2 + 5(G^2 - L^2) \cos(2g))}{4G^2}.
\]
The equations (6) for our system are the averaged equations of the Hamiltonian system with Hamiltonian (8)
\[
\frac{dG}{dl} = \frac{5(1 + 2h^*G^2)(G^2 - K^2) \sin(2g)}{2\sqrt{-2h^*G^2}} = -\varepsilon f_1(g, G, K),
\]
\[
\frac{dg}{dl} = -\varepsilon \frac{6G^4h^* + 5K^2 - 5(2G^4h^* + K^2) \cos(2g)}{2\sqrt{-2h^*G^3}} = \varepsilon f_2(g, G, K),
\]
\[
\frac{dk}{dl} = \frac{K(5(1 + 2h^*G^2) \cos(2g) - 5 - 6h^*G^2)}{2\sqrt{-2h^*G^2}} = \varepsilon f_3(g, G, K),
\]
here \(L = 1/\sqrt{-2h^*} + O(\varepsilon)\), see for more details the proof of Theorem 1. The equilibrium solutions \((g_0, G_0, k^*)\) of this averaged system satisfying (7) give rise to periodic orbits of the Hamiltonian system with Hamiltonian (8) for
each $\mathcal{H} = h^* < 0$ and $K = k^*$, see Theorem 8. There is only one of such

equilibria, namely

$$(g_0, G_0, k^*) = \left( \pm \frac{1}{2} \arccos \left( \frac{3}{5} \right), \frac{1}{\sqrt{-2h^*}}, 0 \right).$$

The signs $\pm$ of $g_0$ provide different initial conditions of the same periodic orbit.

The Jacobian (7) at this equilibrium is equal to $16\sqrt{-2h^*} \neq 0$. So this
equilibrium provides one periodic orbit of the Hamiltonian system with Hamiltonian (8) for each $\mathcal{H} = h^* < 0$ and $K = k^* = 0$. Since $k^* = 0$ from (5) this
periodic orbit bifurcates from a circular orbit of the Kepler problem. Moreover,
since the eigenvalues of the matrix

$$(14) \quad \left( \frac{\partial(f_1, f_2, f_3)}{\partial(g, G, K)} \right)_{(g,G,K)=(g_0,G_0,k^*)}$$
as this equilibrium are $\pm 4$ and $-\sqrt{-2h^*}$, this periodic orbit has a stable manifold of dimension 2 and an unstable of dimension 1. This completes the proof of Theorem 3.

4. THE ANISOtRcopic KEPLER PROBLEM

Computing $\mathcal{P}(E, g, h, G, K)$ we obtain that it is equal to

$$-\frac{(G^2 - K^2) \sin^2 E \cos^2 g}{4L^4(\cos E - 1)^3} + \frac{(G^2 - K^2)(e - \cos E)^2 \cos^2 g}{4G^2L^2(e \cos E - 1)^3}$$

$$+ \frac{(G^2 - K^2)(e - \cos E) \sin E \sin g \cos g}{GL^3(e \cos E - 1)^3} + \frac{(G^2 - K^2) \sin^2 E \sin^2 g}{4L^4(e \cos E - 1)^3}$$

$$- \frac{(G^2 - K^2)(e - \cos E)^2 \sin^2 g}{4G^2L^2(e \cos E - 1)^3} + \frac{G^2 + K^2}{4G^2L^2(e \cos E - 1)}$$

Its averaged function with respect to the mean anomaly is

$$\langle \mathcal{P} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(E, g, h, G, K)(1 - e \cos E) dE$$

$$= \frac{- (G^2 + K^2) (G + L) - (G^2 - K^2)(G - L) \cos(2g)}{4G^2L^2(G + L)}.$$

The equations (6) are the averaged equations of the Hamiltonian system

with Hamiltonian (9)

$$\frac{dG}{dt} = \varepsilon \frac{4(1 - G \sqrt{-2h^*})(-h^*)^{5/2}(G^2 - K^2) \sin(2g)}{\sqrt{2G^2(1 + G \sqrt{-2h^*})}} = -\varepsilon f_1(g, G, K),$$

$$\frac{dg}{dt} = \varepsilon \frac{A}{G^3(1 + G \sqrt{-2h^*})^2} = \varepsilon f_2(g, G, K),$$

$$\frac{dk}{dt} = \varepsilon \frac{2(h^*)^2 K(2h^*G - \sqrt{-2h^*} + (\sqrt{-2h^*} + 2h^*G) \cos(2g))}{G^2(1 + G \sqrt{-2h^*})} = \varepsilon f_3(g, G, K),$$

"
where

\[ A = 4(-h^*)^{5/2} \left((1 - 2G\sqrt{-2h^*} + 2G^2h^*)K^2 + (-G^3\sqrt{-2h^*} + K^2 + G\sqrt{-2h^*}K^2 + 2h^*G^2K^2) \cos(2g)\right), \]

here \( L = 1/\sqrt{-2h^*} + O(\varepsilon) \). The equilibrium solutions \((g_0, G_0, k)^*\) of this averaged system satisfying (7) give rise to periodic orbits of the Hamiltonian system with Hamiltonian (9) for each \( H = h^* < 0 \) and \( K = k^* \), see Theorem 8. There is a unique of such equilibria, namely

\[ (g_0, G_0, k)^* = \left( \pm \frac{\pi}{4}, \frac{1}{\sqrt{-2h^*}}, 0 \right). \]

Again the signs \( \pm \) of \( g_0 \) provide different initial conditions of the same periodic orbit.

The Jacobian (7) at this equilibrium is equal to

\[ 16\sqrt{2}(-h^*)^{19/2} \neq 0. \]

So this equilibrium provides one periodic orbit of the Hamiltonian system with Hamiltonian (9) for each \( H = h^* < 0 \) and \( K = k^* = 0 \). Since \( k^* = 0 \) from (5) this periodic orbit bifurcates from a circular orbit of the Kepler problem. Moreover, since the eigenvalues of the matrix (14) at this equilibrium are \( \pm 2(h^*)^3 \) and \( -4\sqrt{2}(-h^*)^{7/2} \), this periodic orbit has a stable manifold of dimension 2 and an unstable of dimension 1. This completes the proof of Theorem 4.

### 5. The Generalized van der Waals Hamiltonian

Computing \( P(E, g, h, G, K) \) we obtain that it is equal to

\[
\begin{align*}
\frac{(\beta^2 G^2 + G^2 + K^2 - K^2\beta^2)(e \cos E - 1)^2 L^4}{2G^2} - \\
\frac{L^4(G^2 - K^2)(\beta^2 - 1)(e - \cos E)^2 \cos^2 g}{2G^2} + \\
\frac{L^4(G^2 - K^2)(\beta^2 - 1)(e - \cos E)^2 \sin^2 g}{2G^2} - \\
\frac{2L^3(G^2 - K^2)(\beta^2 - 1)(e - \cos E) \cos g \sin E \sin g}{G} + \\
\frac{1}{2} L^2(G^2 - K^2)(\beta^2 - 1) \cos^2 g \sin^2 E \\
\frac{1}{2} L^2(G^2 - K^2)(\beta^2 - 1) \sin^2 E \sin^2 g
\end{align*}
\]

Its averaged function with respect to the mean anomaly is

\[
\langle P \rangle = \frac{1}{2\pi} \int_0^{2\pi} P(E, g, h, G, K)(1 - e \cos E) dE = \frac{B}{4G^2},
\]

where \( B = L^2(5(G^2 - K^2)(G^2 - L^2)(\beta^2 - 1) \cos(2g) - (3G^2 - 5L^2)(G^2 + K^2 + (G^2 - K^2)\beta^2)) \).
The equations (6) are the averaged equations of the Hamiltonian system with Hamiltonian (10)

\[
\frac{dG}{dt} = \varepsilon \frac{5(1 + 2h^*G^2)(G^2 - K^2)(\beta^2 - 1)\sin(2g)}{2G^2\sqrt{2h^*}} = -\varepsilon f_1(g, G, K), \\
\frac{dg}{dt} = -\varepsilon \frac{C}{2G^3\sqrt{2h^*}} = \varepsilon f_2(g, G, K), \\
\frac{dk}{dt} = \varepsilon \frac{K(\beta^2 - 1)(-5 - 6h^*G^2 + 5(1 + 2h^*G^2)\cos(2g))}{2G^2\sqrt{2h^*}} = \varepsilon f_3(g, G, K),
\]

where \(C = 5K^2(\beta^2 - 1) + 6h^*G^4(\beta^2 + 1) - 5(2h^*G^4 + K^2)(\beta^2 - 1)\cos(2g)\) here \(L = 1/\sqrt{-2h^*} + O(\varepsilon)\). The equilibrium solutions \((g_0, G_0, k^*)\) of this averaged system satisfying (7) give rise to periodic orbits of the Hamiltonian system with Hamiltonian (10) for each \(H = h^* < 0\) and \(K = k^*\), see Theorem 8. These equilibria \((g_0, G_0, k^*)\) are

\[
\left( \pm \frac{1}{2} \arccos \left( \frac{3(\beta^2 + 1)}{5(\beta^2 - 1)} \right), \frac{1}{\sqrt{-2h^*}}, 0 \right), \left( \pm \frac{\pi}{2}, \frac{1}{2} \sqrt{\frac{5}{-2h^*}}, \pm \frac{1}{4} \sqrt{\frac{5(1 - 4\beta^2)}{-2h^*(1 - \beta^2)}} \right).
\]

The first two equilibria exist if \(3(\beta^2 + 1)/(5(\beta^2 - 1)) \in [-1, 1]\), i.e., if \(\beta \in (-\infty, -2] \cup [-1/2, 1/2] \cup [2, \infty)\). But for these two first equilibria the signs ± of \(g_0\) provide different initial conditions of the same periodic orbit.

The Jacobian (7) of the first equilibrium is equal to \(J = 16\sqrt{-2h^*}(\beta^2 - 1)(\beta^2 - 4)(\beta^2 - 4/\beta^2 - 1)\). So this equilibrium when \(\beta \in (-\infty, -2) \cup (-1/2, 1/2] \cup (2, \infty)\) provides one periodic orbit of the Hamiltonian system with Hamiltonian (10) for each \(H = h^* < 0\) and \(K = k^* = 0\). Since \(k^* = 0\) from (5) this periodic orbit bifurcates from a circular orbit of the Kepler problem. Moreover, since the eigenvalues of the matrix (14) at this equilibrium are \(\pm 2\sqrt{(\beta^2 - 4)(4\beta^2 - 1)}\) and \(\sqrt{-2h^*}(\beta^2 - 1)\), this periodic orbit has a stable manifold of dimension 2 and an unstable of dimension 1 if \(\beta \in (-1/2, 1/2)\), and has a stable manifold of dimension 1 and an unstable of dimension 2 if \(\beta \in (-\infty, -2) \cup (2, \infty)\). This proves statement (a) of the theorem.

The last four equilibria exist if \(\beta \in (-1, -1/2] \cup [1/2, 1)\) and have Jacobian equal to \(J = -15\sqrt{-2h^*}(\beta^2 - 1)(4\beta^2 - 1)\). So, these four equilibria when \(\beta \in (-1, -1/2) \cup (1/2, 1)\), provide four periodic orbits of the Hamiltonian system with Hamiltonian (10) for each \(H = h^* < 0\) and \(K = k^* = \pm \frac{1}{4} \sqrt{\frac{5(1 - 4\beta^2)}{-2h^*(1 - \beta^2)}} \neq 0\). Since \(k^* \neq 0\) from (5) these periodic orbits bifurcate from elliptic orbits of the Kepler problem. This proves statement (b) of the theorem.

**APPENDIX**

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.
The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [16].

Consider the differential equation
\begin{equation}
\dot{x} = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \quad x(0) = x_0,
\end{equation}
with \( x \in D \) where \( D \) is an open subset of \( \mathbb{R}^n \), and \( t \geq 0 \). Moreover we assume that \( F(t, x) \) is \( T \) periodic in \( t \). Separately we consider in \( D \) the averaged differential equation
\begin{equation}
\dot{y} = \varepsilon f(y), \quad y(0) = x_0,
\end{equation}
where
\[ f(y) = \frac{1}{T} \int_0^T F(t, y) dt. \]

Under certain conditions, see the next theorem, equilibrium solutions of the averaged equation turn out to correspond with \( T \)-periodic solutions of equation (16).

**Theorem 8.** Consider the two initial value problems (15) and (16). Suppose:

(i) \( F \), its Jacobian \( \partial F / \partial x \), its Hessian \( \partial^2 F / \partial x^2 \) are defined, continuous and bounded by an independent constant \( \varepsilon \) in \([0, \infty) \times D\) and \( \varepsilon \in (0, \varepsilon_0] \).
(ii) \( F \) is \( T \)-periodic in \( t \) (\( T \) independent of \( \varepsilon \)).
(iii) \( y(t) \) belongs to \( D \) on the interval of time \([0, 1/\varepsilon]\).

Then the following statements hold.

(a) For \( t \in [0, 1/\varepsilon] \) we have that \( x(t) - y(t) = O(\varepsilon) \), as \( \varepsilon \to 0 \).
(b) If \( p \) is a equilibrium point of the averaged equation (16) and
\begin{equation}
\det \left( \frac{\partial f}{\partial y} \right) \bigg|_{y=p} \neq 0,
\end{equation}
then there exists a \( T \)-periodic solution \( \varphi(t, \varepsilon) \) of equation (15) such that \( \varphi(0, \varepsilon) \to p \) as \( \varepsilon \to 0 \).
(c) The stability or instability of the periodic solution \( \varphi(t, \varepsilon) \) is given by the stability or instability of the equilibrium point \( p \) of the averaged system (16). In fact, the equilibrium point \( p \) has the stability behavior of the Poincaré map associated to the periodic solution \( \varphi(t, \varepsilon) \).

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References


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