A FIRST ORDER AUTOMATED LIE TRANSFORM

ELBAZ I. ABOUELMAGD\textsuperscript{1,2,3}, A. MOSTAFA \textsuperscript{4} AND JUAN L.G. GUIRAO\textsuperscript{5}

Abstract. The objective of the present paper is to contribute to the problem of the normalization of a Hamiltonian system via the elimination of the angle variables involved using the Lie transform technique. The algorithm that we construct assumes that the Hamiltonian is periodic in \( n \) angle variables, with two rates: fast and slow. If the angle variables have the same rate only one transformation is required. The equations needed to evaluate the elements of each transformation and the secular perturbations are constructed.

1. Introduction

Lie transforms perturbation technique is one of the most effective and systematic tool for solving canonical systems of differential equations. The literature is plenty of works on Lie transforms either with a theoretical approach or with a more applied emphasis solving problems from celestial mechanic and differential equations fields.

From our point of view the three main papers contributing the develop of the theory of canonical transformations based on Lie series; in the sense that they make the applications more feasible, practical and systematic; are: Hori [14] where some formulas for the Lie transforms in the form of canonical invariance that can work for any set of canonical variables are given, Deprit [6] extends the theory to cover the cases where the generating function depends explicitly on a small parameter and Kamel [15] gives recurrence formulas for Deprit’s equations which made the use of Lie transforms easier and applicable for computerized symbolic calculations.


Key words and phrases. Dynamical Systems, Hamiltonian Systems, Averaging Theory.

2010 Mathematics Subject Classification. Primary: 70E17, 70E20, 70E40. Secondary: 37C27.

Lie transforms technique has been extensively applied for studying the motion of artificial and natural bodies in celestial mechanics. In particular the literature is plenty of works where this technique is used to study the translational motion of artificial satellites and the coupled translational–rotational motion.

One of the first applications of the method in the artificial satellite theory was presented by Deprit and Rom [7] using it as theoretical foundation for programming the analytical solution of the main problem in satellite theory.


Bernard De Saedeleer [28] studies the zonal problem of the artificial satellite using the Lie Transform as canonical perturbation technique. An analytical formula to compute directly the first order averaged Hamiltonian in closed form is given.

Bernard De Saedeleer [29] uses the Lie Transform to achieve an analytical theory of a Lunar artificial satellite considering four main perturbations: the synchronous rotation of the Moon, the oblateness of the Moon, the triaxiality the Moon and the major third body effect of the Earth.

Tupikova [32] gives a mathematical approach for constructing an averaging procedure for the motion of a massless body around the central body perturbed by fully interacting planets. Carvalho et al. [4] applies Lie–Hori perturbation method up to the second–order to present an analytical theory with numerical simulations to study the orbital motion of lunar artificial satellites considering the perturbations of the non–uniform distribution of mass of the Moon and of a third–body in elliptical orbit. Liu et al. [24] investigated quasi–circular frozen orbits in the Martina gravity field analytically using two different methods: first using a Lagrangian formulation, and second by applying Lie transforms. By comparison, the two methods were found well consistent with each other.

One of the first and most important works that used Lie series to study the translational rotational motion of satellites was presented by Kinoshita
in 1972 who used the Hori–Lie transformation to give a complete first order treatment of the translational rotational motion of a triaxial satellite when the central body is a sphere. Elpe and Vallejo [9] studies the attitude dynamics of perturbed triaxial rigid bodies and a Lie transformation is used for solving the resulting equations. R.V. de Moraes et al. [26] analyzes the stability of the rotational motion of artificial satellites perturbed by the gravity gradient torque. A semi–analytic process of normalization based on Lie–Hori algorithm is applied to obtain the Hamiltonian normalized up to the fourth order.

Lie transforms technique has been used effectively in the restricted 3–body problem, for instance Delva [8] uses the method of Lie series to construct a solution for the elliptic restricted three body problem. Coppola and Rand [5] investigated the nonlinear stability of $L_4$ in the circular restricted three–body problem. Hamdy et al. [11] develops an explicit analytical expressions for the small amplitude orbits of infinitesimal mass on the equilibrium points in the elliptic restricted three body problem. Libert and Henrard [23] uses Lie transformations to find the four fundamental frequencies of the 3–D secular three–body problem and the long–term time evolutions of the Keplerian elements were computed. While in recent days Abouelmagd et al. [1] gave a numerical integration for the equations of motion of an infinitesimal body in the restricted three-body problem under the effect of the zonal harmonic parameters of the bigger primary up to $J_4$ via a Lie series approach. Furthermore, they presented an algorithm that allows us to find any number of Lie series terms and which gives successful calculations for the orbit of the infinitesimal body around one of the libration points. Finally, they applied the obtained result to the case of the Earth–Moon system.

Lie transforms perturbation technique has been also applied in the theory of the natural heavenly bodies but with a less frequency. Some of the more relevant papers on this topic could be: Message [25] where some new results in the study of the orbit of Saturn’s satellite Hyperion are presented using Lie series method. Segerman and Richardson [30] constructs a complete analytical dynamic theory for the motion of Nereid. Kamel [17] and [18] present the complete solution of the planetary canonical equations of motion by the method of G. Hori through successive changes of canonical variables using the Lie series. Kamel and Soliman [20] extends the construction of the Jupiter–Saturn theory to include all the terms up to the seventh order in the masses.

It is natural that Lie transform method be applied to the theory of differential equations, e.g. Kamel [16] shows the equivalence of solving the perturbed non–Hamiltonian system $\dot{y} = g(y, t; \varepsilon)$ where the Hamiltonian system is represented by the Hamiltonian $K = Y \cdot g(y, t; \varepsilon)$, where $Y$ is the adjoint vector which allows the direct use of the perturbation methods already established for Hamiltonian systems. Pramana [27] investigates the
three–dimensional non–Hermitian systems using classical perturbation theory based on Lie transformations. It was found that semiclassical energy eigenvalues calculated with the classical perturbation theory are in very good agreement with exact energies.

In this work a complete first order solution is given for any Hamiltonian problem which satisfies the conditions for applying Lie transforms. The Hamiltonian is assumed periodic in \( n \) angles with two rates: fast and slow. The automated identities are given for both the short and long period transformations required for the elimination of the fast and slow angles respectively. The equations of the elements and the generating functions of each transformation are constructed as well as the secular perturbations. The procedure can be extended systematically to any required higher order.

2. Lie Transformations

Lie perturbation technique works through elimination of the angular variables from the Hamiltonian through successive transformations according to their rates of change. If all the involved angles are of different rates (fast, intermediate and slow), then three transformations should be performed. If two or more angles have the same rate of change, then they are eliminated through the same transformation. Let the Hamiltonian be presented as

\[
F(l_i, L_i) = F_0(L_i) + \sum_{k \geq 1} F_k(l_i, L_i)
\]

where \( l_i' \)'s stand for the angular variables and \( L_i' \)'s stand for the momenta.

The idea is to eliminate the angular variables of the higher orders parts of the Hamiltonian, \( F_k, k \geq 1 \) to obtain a new Hamiltonian containing only the action variables having an integrable system of equations up to the desired order. This would be done through successive transformations \((l_i, L_i) \rightarrow (x_i, X_i) \rightarrow \ldots \) to end with a new set of variables say \((z_i, Z_i)\) and a new Hamiltonian of the form \( F^*(Z_i) \), where small letters represent the angles and capital letters represent the action variables.

In what follows, we list the identities of Lie transformation (see Deprit [6] and Kamel [15]) for elimination of one variable (say \( l_j \)). In case of elimination of two or more variables at the same time, the same procedure is used with averaging over all the variables to be eliminated.

The generating function of the transformation will have the form,

\[
w(l_i, L_i) = \sum_{n \geq 1} w_n(l_i, L_i),
\]

where \( w_n \) is of order \( n \).

The transformed Hamiltonian is found via the relations,

\[
F_0^* = F_0
\]
and

\[ F^*_n = \frac{1}{2\pi} \int_0^{2\pi} \tilde{F}_n dl_j \]

where

\[ \tilde{F}_n = F_n + \sum_{j=1}^{n-1} C_{j-1}^{n-1}(F_{n-j}, w_j) + C_j^{n-1}G_jF^*_{n-j} \]

being

\[ G_j = L_j - \sum_{m=0}^{j-2} C_{j-1}^{j-1}L_{m+1}G_{j-m-1}, \]

where \( G_1 = L_1 \).

\( L_n \) is the Lie derivative of order \( n \) defined by,

\[ L_n f = (f, w_n), \]

where \((f, g)\) is the Poisson bracket of the two functions \( f(l_i, L_i) \) and \( g(l_i, L_i) \) defined by

\[ (f, g) = \sum_{i=1}^{n} \frac{\partial f}{\partial l_i} \frac{\partial g}{\partial L_i} - \frac{\partial f}{\partial L_i} \frac{\partial g}{\partial l_i}. \]

The generating function of order \( n \), \( w_n \), is found by solving the linear first order partial differential equation

\[ (F_0, w_n) = F^*_n - \tilde{F}_n. \]

The relation between the old and new elements are found through the relations,

\[ l_i = x_i + \sum_{n \geq 1} x_i^{(n)} \]

\[ L_i = X_i + \sum_{n \geq 1} X_i^{(n)} \]

where

\[ x_i^{(n)} = \frac{\partial w_n}{\partial X_i} + \sum_{j=1}^{n-1} C_{j-1}^{n-1}G_jx_i^{n-j} \]

\[ X_i^{(n)} = -\frac{\partial w_n}{\partial x_i} + \sum_{j=1}^{n-1} C_{j-1}^{n-1}G_jX_i^{n-j}. \]

The inverse transformation is given by,

\[ x_i = l_i + \sum_{n \geq 1} l_i^{(n)} \]

\[ X_i = L_i + \sum_{n \geq 1} L_i^{(n)} \]
where
\[ l_i^{(n)} = -x_i^n + \sum_{j=1}^{n-1} C_j^n G_j x_i^{n-j} \]
\[ L_i^{(n)} = -X_i^n + \sum_{j=1}^{n-1} C_j^n G_j X_i^{n-j}. \]

2.1. First order short period transformation identities. In this section, the identities of the first order short period transformation will be listed illustrating the elimination of the fast variable \( l_j \) from the perturbed function \( F_1(l_i, L_i) \) through the short period transformation \( (l_i, L_i) \rightarrow (x_i, X_i) \),

\[
F_0 = F_0(L_j) \\
F_1 = F_1(l_i, L_i)
\]

We seek a new first order Hamiltonian \( F_1^* \) free from the fast variable \( l_j \). This is done through the transformation,

\[
F_1(l_i, L_j) \rightarrow F_1^*(x_{i \neq j}, X_i).
\]

The transformed Hamiltonian \( F_1^* \) is given by:

\[
(1) \\
F_1^* = \frac{1}{2\pi} \int_0^{2\pi} F_1 dl_j.
\]

The first order relation between the old and new sets of variables are found from

\[
x_i = l_i + \frac{\partial w_1}{\partial L_i}, \\
X_i = L_i - \frac{\partial w_1}{\partial l_i}.
\]

Where the first order generating function \( w_1 \) is found by solving the first order linear partial differential equation,

\[
(F_0, w_1) = F_1^* - F_1
\]

which gives us,

\[
\frac{\partial F_0}{\partial L_j} \frac{\partial w_1}{\partial l_j} = F_1 - F_1^* = P_1.
\]

2.2. First order short period automated lie transform. In what follows, we present a first order automated Lie transform for a Hamiltonian with \( 2n \) variables \( l_i \) and \( L_i \), \( 1 \leq i \leq n \) in the form

\[
F(l_i, L_i) = F_0(L_i) + F_1(l_i, L_i)
\]
where,

\[ F_1(l_i, L_i) = \sum_{a_1, \ldots, a_n} F_{a_1, \ldots, a_n}^{1C}(L_i) \cos \left( \sum_{k=1}^{n} a_k l_k \right) + \sum_{b_1, \ldots, b_n} F_{b_1, \ldots, b_n}^{1S}(L_i) \sin \left( \sum_{k=1}^{n} b_k l_k \right). \]

When the Hamiltonian contains only two rates: fast and slow, we rename the fast angles to be \( l_n, l_{n-1}, \ldots, l_{n-r}, l_{n-r+1} \). Consequently the slow angles will be \( l_{n-r}, l_{n-r-1}, \ldots, l_1 \).

In this formulation, the transformation identities for the elimination of the \( r \) fast variables are presented in the form,

\[ \begin{align*}
F_0 &= F_0(L_n, L_{n-1}, \ldots, L_{n-r}, L_{n-r+1}), \\
F_1 &= F_1(l_i, L_i).
\end{align*} \]

The transformation of \( F_1 \) will be such that

\[ F_1(l_i, L_j) \to F_1^*(x_1, x_2, \ldots, x_{n-r}, X_i). \]

Then the effect of the averaging process in (1) will transform \( F_1(l_i, L_j) \) into \( F_1^*(x_1, x_2, \ldots, x_{n-r}, X_i) \) which will have the form,

\[ (2) \quad F_1^*(x_1, x_2, \ldots, x_{n-r}, X_i) = \sum_{a_1, \ldots, a_{n-r}} F_{a_1, \ldots, a_{n-r}, 0, \ldots, 0}^{1C}(X_i) \cos \left( \sum_{k=1}^{n-r} a_k x_k \right) + \sum_{b_1, \ldots, b_n} F_{b_1, \ldots, b_n, 0, \ldots, 0}^{1S}(X_i) \sin \left( \sum_{k=1}^{n-r} b_k x_k \right) \]

where \( F_{a_1, \ldots, a_n}^{1C} = F_{a_1, \ldots, a_n}^{1C} \) and \( F_{b_1, \ldots, b_n}^{1S} = F_{b_1, \ldots, b_n}^{1S} \) for \( X_i = L_i \).

The condition for applying the Lie’s technique is to have \( F_1^* \) free of angles. This is satisfied by,

\[ (3) \quad F_{a_1, \ldots, a_n, 0, \ldots, 0}^{1C} = F_{b_1, \ldots, b_n, 0, \ldots, 0}^{1S} = 0 \]

for all \( a_k \neq 0 \) and \( b_k \neq 0 \) simultaneously. This simply means that fast variables appear in \( F_1 \) only in conjunction with the slow variables.

Thus, we conclude that

\[ F_1^* = F_{0, \ldots, 0}^{1C}. \]

The periodic part \( P_1 \) will be then given by,

\[ P_1(l_i, L_i) = \sum_{a_1, \ldots, a_n} P_{a_1, \ldots, a_n}^{C} \cos \left( \sum_{k=1}^{n} a_k l_k \right) + \sum_{b_1, \ldots, b_n} P_{b_1, \ldots, b_n}^{S} \sin \left( \sum_{k=1}^{n} b_k l_k \right) \]

where \( P_{a_1, \ldots, a_n}^{C} = P_{a_1, \ldots, a_n}^{C} \) for all values except \( P_{0, \ldots, 0}^{C} = 0 \) and \( P_{b_1, \ldots, b_n}^{S} = F_{b_1, \ldots, b_n}^{S} \). The generating function \( w_1 \) will be found from the linear first order PDE.
\[
\sum_{k=n-r+1}^{n} \frac{\partial F_0}{\partial L_k} \frac{\partial w_1}{\partial l_k} = \sum_{a_1,\ldots,a_n} P^C_{a_1,\ldots,a_n} \cos \left( \sum_{k=1}^{n} a_k l_k \right) + \sum_{b_1,\ldots,b_n} P^S_{b_1,\ldots,b_n} \sin \left( \sum_{k=1}^{n} b_k l_k \right)
\]

which has the solution,

\[
w_1(l_i, L_i) = \sum_{b_1,\ldots,b_n} w^1_{b_1,\ldots,b_n} \cos \left( \sum_{k=1}^{n} b_k l_k \right) + \sum_{a_1,\ldots,a_n} w^1_{a_1,\ldots,a_n} \sin \left( \sum_{k=1}^{n} a_k l_k \right)
\]

with

\[
w^1_{a_1,\ldots,a_n} = \frac{P^1C_{a_1,\ldots,a_n}}{\sum_{k=n-r+1}^{n} a_k \frac{\partial F_0}{\partial L_k}},
\]

\[
w^1_{b_1,\ldots,b_n} = -\frac{P^1S_{b_1,\ldots,b_n}}{\sum_{k=n-r+1}^{n} b_k \frac{\partial F_0}{\partial L_k}}.
\]

Using (3) and \( P^C_{a_1,\ldots,a_n} = F^C_{a_1,\ldots,a_n} \) for all values except \( P^C_{0,\ldots,0} = 0 \) and \( P^S_{b_1,\ldots,b_n} = F^S_{b_1,\ldots,b_n} \) we get

\[
w^1_{a_1,\ldots,a_n} = \frac{F^1C_{a_1,\ldots,a_n}}{\sum_{k=n-r+1}^{n} a_k \frac{\partial F_0}{\partial L_k}}, \quad w^1_{a_1,\ldots,a_n-1,0,\ldots,0} = 0,
\]

\[
w^1_{b_1,\ldots,b_n} = -\frac{F^1S_{b_1,\ldots,b_n}}{\sum_{k=n-r+1}^{n} b_k \frac{\partial F_0}{\partial L_k}}, \quad w^1_{b_1,\ldots,b_n-1,0,\ldots,0} = 0.
\]

The first order relation between the old and new sets of variables is given by

\[
x_i = l_i + \frac{\partial w_1}{\partial L_i},
\]

\[
X_i = L_i - \frac{\partial w_1}{\partial l_i}
\]

where \( 1 \leq i \leq n \).

3. Transformation identities

3.1. First order long period transformation identities. Before we get a first order long period transformation, we should get the second order short period transformed Hamiltonian \( F^*_2(x_1, \ldots, x_{n-r}, -, X_i) \). The corresponding identities of the second order short period Lie transformation are

\[
\tilde{F}_2 = (F_1, w_1) + (F^*_1, w_1).
\]

The averaged Hamiltonian is given by
A FIRST ORDER AUTOMATED LIE TRANSFORM

\[ F_2^* = \left( \frac{1}{2\pi} \right)^r \int_0^{2\pi} \cdots \int_0^{2\pi} \bar{F}_2 dl_n dl_{n-1} \cdots dl_{n-r+1}. \]

Now, we eliminate the slow angles \( x_i, \ 1 \leq i \leq n - r \) through the long period transformation, \( x_i, X_i \to y_i, Y_i \), where the basic identities are,

\[
\begin{align*}
F_0^*(Y_i) &= F_0^*(X_i)|_{X_i \to Y_i}, \\
F_1^*(Y_i) &= F_1^*(X_i)|_{X_i \to Y_i}, \\
F_2^* &= F_2^*(x_1, \ldots, x_{n-r}, -, X_i), \\
F_2^*(x_1, \ldots, x_{n-r}, -, X_i) &\to F_2^*(-, Y_i).
\end{align*}
\]

where \( F_{2}^{**} \) is obtained by averaging \( F_{2}^{*} \) over \( x_{n-r}, x_{n-r-1}, \ldots, x_{1} \).

\[
F_{2}^{**} = \left( \frac{1}{2\pi} \right)^{n-r} \int_0^{2\pi} \cdots \int_0^{2\pi} F_2^* dx_{n-r} dx_{n-r-1} \cdots dx_1.
\]

The periodic part is given by,

\[
P_2 = F_2^* - F_{2}^{**}.
\]  

Then the generating function \( w_1^* \) will be found from the first order linear PDE

\[
\sum_{k=1}^{n-r} \partial F_1^* \frac{\partial w_1^*}{\partial x_k} = P_2.
\]

The elements of the long period transformation are given by

\[
\begin{align*}
y_i &= x_i + \frac{\partial w_1^*}{\partial X_i}, \\
Y_i &= X_i - \frac{\partial w_1^*}{\partial x_i}
\end{align*}
\]

where \( 1 \leq i \leq n. \)

3.2. First order long period automated Lie transform. Let \( F_1^* \) be stated in (2) the required Poisson brackets in (5) are computed. After simplification, \( F_2 \) can be expressed in the form

\[
\bar{F}_2(I_i, L_i) = \sum_{a_1, \ldots, a_n} F_{2}^{2C}_{a_1, \ldots, a_n} \cos \left( \sum_{k=1}^{n} a_k I_k \right) + \sum_{b_1, \ldots, b_n} F_{2}^{2S}_{b_1, \ldots, b_n} \sin \left( \sum_{k=1}^{n} b_k I_k \right)
\]

where the coefficients are given as follows:

\[
F_{m_1, \ldots, m_n}^{2C} = B_{m_1, \ldots, m_n}^{1C} + B_{m_1, \ldots, m_n}^{2C}, \quad F_{m_1, \ldots, m_n}^{2S} = B_{m_1, \ldots, m_n}^{1S} + B_{m_1, \ldots, m_n}^{2S}
\]
\[ B_{m_1,\ldots,m_n}^{1C} = \frac{1}{2} \sum_{i=1}^{n} \left[ \sum_{j=1, r_j-t_j=m_j} r_i \left( H_{r_1,\ldots,r_n}^{1C} \frac{\partial w_{t_1,\ldots,t_n}^{1S}}{\partial L_i} + H_{r_1,\ldots,r_n}^{1S} \frac{\partial w_{t_1,\ldots,t_n}^{1C}}{\partial L_i} \right) \right. \]
\[ - t_i \left( w_{t_1,\ldots,t_n}^{1S} \frac{\partial H_{r_1,\ldots,r_n}^{1C}}{\partial L_i} + w_{t_1,\ldots,t_n}^{1C} \frac{\partial H_{r_1,\ldots,r_n}^{1S}}{\partial L_i} \right) \]
\[ + \left. \sum_{j=1, r_j-t_j=m_j} r_i \left( H_{r_1,\ldots,r_n}^{1S} \frac{\partial w_{t_1,\ldots,t_n}^{1C}}{\partial L_i} - H_{r_1,\ldots,r_n}^{1C} \frac{\partial w_{t_1,\ldots,t_n}^{1S}}{\partial L_i} \right) \right]. \]

\[ B_{m_1,m_2,m_3}^{1S} = \frac{1}{2} \sum_{i=1}^{n} \left[ \sum_{j=1, r_j+t_j=m_j} r_i \left( H_{r_1,\ldots,r_n}^{1S} \frac{\partial w_{t_1,\ldots,t_n}^{1C}}{\partial L_i} - H_{r_1,\ldots,r_n}^{1C} \frac{\partial w_{t_1,\ldots,t_n}^{1S}}{\partial L_i} \right) \right. \]
\[ + t_i \left( w_{t_1,\ldots,t_n}^{1C} \frac{\partial H_{r_1,\ldots,r_n}^{1S}}{\partial L_i} - w_{t_1,\ldots,t_n}^{1S} \frac{\partial H_{r_1,\ldots,r_n}^{1C}}{\partial L_i} \right) \]
\[ - \left. \sum_{j=1, r_j-t_j=m_j} r_i \left( H_{r_1,\ldots,r_n}^{1S} \frac{\partial w_{t_1,\ldots,t_n}^{1C}}{\partial L_i} + H_{r_1,\ldots,r_n}^{1C} \frac{\partial w_{t_1,\ldots,t_n}^{1S}}{\partial L_i} \right) \right]. \]

\[ B_{m_1,\ldots,m_n}^{2C} = - \sum_{i=1}^{n} t_i \frac{\partial H_{r_1,\ldots,r_n}^{1C}}{\partial L_i} w_{m_1,\ldots,m_n}^{1S}, \quad B_{m_1,\ldots,m_n}^{2S} = \sum_{i=1}^{n} t_i \frac{\partial H_{r_1,\ldots,r_n}^{1C}}{\partial L_i} w_{m_1,\ldots,m_n}^{1C}. \]

Performing the averaging over \([l_n, l_{n-1}, \ldots, l_{n-r+1}]\) on (9) and using first equality of (4) we get

\[ 10 \]

\[ F_2^* = \sum_{a_1,\ldots,a_{n-r}} F_{a_1,\ldots,a_{n-r},0,\ldots,0}(X_i) \cos \left( \sum_{k=1}^{n-r} a_k x_k \right) + \sum_{b_1,\ldots,b_{n-r}} F_{b_1,\ldots,b_{n-r},0,\ldots,0}(X_i) \sin \left( \sum_{k=1}^{n} b_k \right) \]

Averaging now (10) over \([x_{n-r}, x_{n-r-1}, \ldots, x_1]\), we get

\[ F_2^{**} = F_{0,\ldots,0}^{2C}. \]

(6) will give then

\[ P_2 = \sum_{a_1,\ldots,a_{n-r}} P_{a_1,\ldots,a_{n-r}}^{2C} \cos \left( \sum_{k=1}^{n-r} a_k x_k \right) + \sum_{b_1,\ldots,b_{n-r}} P_{b_1,\ldots,b_{n-r}}^{2S} \sin \left( \sum_{k=1}^{n} b_k x_k \right) \]

with \(P_{a_1,\ldots,a_{n-r}}^{2C} = F_{a_1,\ldots,a_{n-r},0,\ldots,0}^{2C}\) for all values except \(P_{0,\ldots,0}^{2C} = 0\) and \(P_{b_1,\ldots,b_{n-r}}^{2S} = F_{b_1,\ldots,b_{n-r},0,\ldots,0}^{2S}\).

Now, (7) is particularized as
\[ \sum_{k=1}^{n-r} \frac{\partial F_i^*}{\partial x_k} \frac{\partial w_i^*}{\partial x_k} = \sum_{a_1, \ldots, a_{n-r}} P_{a_1, \ldots, a_{n-r}}^2 C \cos \left( \sum_{k=1}^{n-r} a_k x_k \right) + \sum_{b_1, \ldots, b_{n-r}} P_{b_1, \ldots, b_{n-r}}^2 S \sin \left( \sum_{k=1}^{n-r} b_k x_k \right) \]

which has the solution,

\[ w_1^* = \sum_{b_1, \ldots, b_{n-r}} w_{b_1, \ldots, b_{n-r}}^1 C \cos \left( \sum_{k=1}^{n-r} b_k x_k \right) + \sum_{a_1, \ldots, a_{n-r}} w_{a_1, \ldots, a_{n-r}}^1 S \sin \left( \sum_{k=1}^{n-r} a_k x_k \right) \]

where the coefficients are given by

\[ w_{a_1, \ldots, a_{n-r}}^1 S = \frac{F_{a_1, \ldots, a_{n-r}, 0, \ldots, 0}^{2C}}{\sum_{k=1}^{n-r} a_k \frac{\partial F_i^*}{\partial X_k}}, \quad w_{0, \ldots, 0}^1 S = 0, \]

\[ w_{b_1, \ldots, b_{n-r}, 0, \ldots, 0}^1 C = \frac{-F_{b_1, \ldots, b_{n-r}, 0, \ldots, 0}^{2S}}{\sum_{k=1}^{n-r} b_k \frac{\partial F_i^*}{\partial X_k}}, \quad w_{0, \ldots, 0}^1 C = 0. \]

The elements of the long period transformation are given by (8).

4. **Secular perturbations**

The equations of motion are now reduced to

\[ \frac{dY_i}{dt} = \frac{\partial F^{**}}{\partial y_i} = 0, \]
\[ \frac{dy_i}{dt} = \frac{\partial F^{**}}{\partial Y_i} = C_i, \]
\[ F^{**} = F_0^{**} + F_1^{**}, \]

where \( 1 \leq i \leq n \) and \( C_i \) are constants determined in terms of \( Y'_i \)’s.

Note that the previous system admits the solution:

\[ Y_i = Y_{i,0}, \]
\[ y_i = y_{i,0} + C_i t \]

where \( 1 \leq i \leq n \) and the constants \( C_i \) are obtained from the initial conditions of the original variables through the equations of transformations of the elements (4) and (8) respectively.

Having determined \( y_i \) and \( Y_i \) we can evaluate \( F^{***} = F^{**}(Y_i) \) and determine the constants \( C'_i \)’s. Finally, we use (8) to get \( x_i \) and \( X_i \). Having \( x_i(t) \) and \( X_i(t) \), (4) are applied to obtain the original elements \( (l_i, L_i) \) evaluated at any time \( t \).
5. Appendix: Procedure summary

In this section we summarize the procedure technique previously developed.

Given the Hamiltonian system,

\[
\frac{d}{dt} L_i = -\frac{\partial F}{\partial l_i}, \\
\frac{d}{dt} l_i = \frac{\partial F}{\partial L_i},
\]

\(1 \leq i \leq n\) with initial state \((l_0, L_0)\) and Hamiltonian given by

\[
F(I_i, L_i) = F_0(L_{n-r+1}, L_{n-r+2}, \ldots, L_n) + \sum_{i_1, \ldots, i_n} F_{i_1, \ldots, i_n}^C \cos \left( \sum_{k=1}^{n} i_k l_k \right) + F_{i_1, \ldots, i_n}^S \sin \left( \sum_{k=1}^{n} i_k l_k \right)
\]

with \(F_{i_1, \ldots, i_{n-r+1},0,\ldots,0}^C = F_{i_1, \ldots, i_{n-r+1},0,\ldots,0}^S = 0\) for all \(i_k \neq 0\) simultaneously.

The solution is given by \((l_i(t), L_i(t))\) where

\[
l_i = x_i + \frac{\partial w_1}{\partial L_i}
\]

\[
L_i = X_i - \frac{\partial w_1}{\partial l_i}
\]

where \(1 \leq i \leq n\) and

\[
w_1(l_i, L_i) = \sum_{i_1, \ldots, i_n} w_{i_1, \ldots, i_n}^{1C} \cos \left( \sum_{k=1}^{n} i_k l_k \right) + w_{i_1, \ldots, i_n}^{1S} \sin \left( \sum_{k=1}^{n} i_k l_k \right)
\]

with

\[
w_{i_1, \ldots, i_n}^{1S} = \frac{F_{i_1, \ldots, i_n}^{1S}}{\sum_{k=1}^{n} i_k \frac{\partial F_0}{\partial L_k}}, \quad w_{i_1, \ldots, i_n}^{1S} = \frac{F_{i_1, \ldots, i_n}^{1S}}{\sum_{k=1}^{n} i_k \frac{\partial F_0}{\partial L_k}}
\]

Note that

\[
x_i = y_i - \frac{\partial w_1^*}{\partial X_i}, \quad X_i = Y_i + \frac{\partial w_1^*}{\partial x_i}
\]

where \(1 \leq i \leq n\) and
where

\[ F_{m_1,...,m_n} = B_{m_1,...,m_n}^{1C} + B_{m_1,...,m_n}^{2C}, ~ F_{m_1,...,m_n} = B_{m_1,...,m_n}^{1S} + B_{m_1,...,m_n}^{2S} \]

and

\[ B_{m_1,...,m_n}^{1C} = \frac{1}{2} \sum_{i=1}^{n} \left[ \sum_{j=1, r_j - t_j = m_j} r_i \left( H_{r_1,...,r_n}^{1C} \frac{\partial w_{l_1,...,l_n}^{1S}}{\partial L_i} + H_{r_1,...,r_n}^{1S} \frac{\partial w_{l_1,...,l_n}^{1C}}{\partial L_i} - H_{r_1,...,r_n}^{1C} \frac{\partial w_{l_1,...,l_n}^{1S}}{\partial L_i} - H_{r_1,...,r_n}^{1C} \frac{\partial w_{l_1,...,l_n}^{1C}}{\partial L_i} \right) \right. \\
\left. - t_i \left( w_{l_1,...,l_n}^{1S} \frac{\partial H_{r_1,...,r_n}^{1C}}{\partial L_i} + w_{l_1,...,l_n}^{1C} \frac{\partial H_{r_1,...,r_n}^{1S}}{\partial L_i} \right) + t_i \left( w_{l_1,...,l_n}^{1C} \frac{\partial H_{r_1,...,r_n}^{1S}}{\partial L_i} + w_{l_1,...,l_n}^{1S} \frac{\partial H_{r_1,...,r_n}^{1C}}{\partial L_i} \right) \right] \]

\[ B_{m_1,...,m_n}^{1S,m_2,m_3} = \frac{1}{2} \sum_{i=1}^{n} \left[ \sum_{j=1, r_j + t_j = m_j} r_i \left( H_{r_1,...,r_n}^{1S} \frac{\partial w_{l_1,...,l_n}^{1S}}{\partial L_i} - H_{r_1,...,r_n}^{1C} \frac{\partial w_{l_1,...,l_n}^{1C}}{\partial L_i} \right) \\
+ t_i \left( w_{l_1,...,l_n}^{1C} \frac{\partial H_{r_1,...,r_n}^{1C}}{\partial L_i} - w_{l_1,...,l_n}^{1S} \frac{\partial H_{r_1,...,r_n}^{1S}}{\partial L_i} \right) \\
- \sum_{j=1, r_j - t_j = m_j} n \left( H_{r_1,...,r_n}^{1S} \frac{\partial w_{l_1,...,l_n}^{1S}}{\partial L_i} + H_{r_1,...,r_n}^{1C} \frac{\partial w_{l_1,...,l_n}^{1C}}{\partial L_i} \right) \\
+ t_i \left( w_{l_1,...,l_n}^{1S} \frac{\partial H_{r_1,...,r_n}^{1C}}{\partial L_i} + w_{l_1,...,l_n}^{1C} \frac{\partial H_{r_1,...,r_n}^{1S}}{\partial L_i} \right) \right] \]

\[ B_{m_1,...,m_n}^{2C} = - \sum_{i=1}^{n} t_i \frac{\partial H_{0,...,0}^{1C}}{\partial L_i} w_{m_1,...,m_n}^{1S}, \quad B_{m_1,...,m_n}^{2S} = \sum_{i=1}^{n} t_i \frac{\partial H_{0,...,0}^{1C}}{\partial L_i} w_{m_1,...,m_n}^{1C}. \]

Now,

\[ Y_i = Y_{i,0}, \]
\[ y_i = y_{i,0} + C_i t \]
where $1 \leq i \leq n$ and for the initial conditions we have
\[
y_i(0) = x_i(0) - \frac{\partial w^*}{\partial X_i}|_{X_i = X_i(0)},
\]
\[
X_i = Y_i + \frac{\partial w^*}{\partial x_i}|_{x_i = x_i(0)},
\]
$1 \leq i \leq n$ and
\[
C_i = \frac{\partial F^{**}(Y_i)}{\partial Y_i},
\]
\[
F^{**}(Y_i) = F_0(L_{n-r+1}, \ldots, L_n)|_{L_i \rightarrow Y_i} + F_1^* (X_1, \ldots, X_n)|_{X_i \rightarrow Y_i};
\]
\[
F^{**}(Y_i) = F_0(L_{n-r+1}, \ldots, L_n)|_{L_i \rightarrow Y_i} + F_1^{**C} (L_1, \ldots, L_n)|_{L_i \rightarrow Y_i}.
\]
Finally, using the transformation equations of the elements, we get
\[
x_i(0) = l_i(0) - \frac{\partial w_1}{\partial L_i}|_{L_i = l_i(0)},
\]
\[
X_i(0) = L_i(0) - \frac{\partial w_1}{\partial l_i}|_{l_i = l_i(0)},
\]
$1 \leq i \leq n$.

6. Conclusion

In this paper an algorithm is generated to solve Hamiltonian systems using the Lie transform technique. The algorithm constructed assumes that the Hamiltonian is periodic in $n$ angle variables, with two rates: fast and slow. The procedure gives a complete first order solution, which can be extended systematically to any required higher order.

Acknowledgements

This work has been partially supported by MICINN/FEDER grant number MTM2011–22587.

References


1 Mathematics Department, Faculty of Science and Arts (Khulais), University of Jeddah, Saudi Arabia.

2 National Research Institute of Astronomy and Geophysics, Cairo, Egypt—Corresponding Author—

3 Nonlinear Analysis and Applied Mathematics Research Group (NAAM), Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

E-mail address: eabouelmagd@gmail.com or eabouelmagd@kau.edu.sa

4 Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

5 Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Hospital de Marina, 30203–Cartagena, Región de Murcia, Spain.

E-mail address: juan.garcia@upct.es