DYNAMICS OF A DUMBELL SATELLITE UNDER THE ZONAL HARMONIC EFFECT OF AN OBLATE BODY

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ABSTRACT. The aim of the present paper is to study the dynamics of a dumbbell satellite moving in a gravity field generated by an oblate body considering the effect of the zonal harmonic parameter. We prove that the pass trajectory of the mass center of the system is periodic and different from the classical one when the effect of the zonal harmonic parameter is non zero. Moreover, we complete the classical theory showing that the equations of motion in the satellite approximation can be reduced to Beletsky’s equation when the zonal harmonic parameter is zero. The main tool for proving these results is the Lindstedt–Poincare’s technique.

1. INTRODUCTION

From the end of the sixth decade of the last century, a part of the mathematical community, has directed its attention to the study the so called dumbbell body or satellite in central gravity, see for instance Morán [13], Schechter [17], Brereton and Modi [3], Beletsky [2, 1], Maciejewski et al. [12]; Kirchgraber et al. [10], Krupa et al. [11], Elipe et al. [7], Burov and Dugain [4] or Nakanishi et al. [15].

Recall that a dumbbell body is a quite simple structure composed by two masses connected by a massless rod. It is assume that this object is moving around a planet whose gravity field is approximated by the field of the attracting center. In general, the distance between the two points masses is considered to be much smaller that the distance between the satellite’s center of mass and the attracting center of mass. Thus, it is common to neglect the influence of the attitude dynamics on the motion of the center of mass and treat it as an unperturbed Keplerian one.

Rodnikov [16] studied equilibrium positions of a weight on a cable fixed to a dumbbell–shaped space station moving along a circular geocentric orbit. This model is composed by two masses coupled by a weightless rod, while the cable is weightless and non-stretched. It is derived the equation of motion when the motion proceeds in a single plane, while the center of mass of...

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the system moves along a circular geocentric orbit. In addition, were found equilibrium configurations of the system and analyzed Lyapunov stability of configurations for two situations, first when the station is composed of equal masses, second when masses at the ends of the station are different.

For the “dumbbells–load” system with two unilateral connections, all relative equilibria on the circular Keplerian orbit were established by Munitsina (2007). These results were interpreted for studying the relative equilibria for which both connections are stretched in geometrical terms. The necessary and sufficient conditions for stability of the relative equilibria were stated.

Celletti and Sidorenko [6] investigated the dumbbell satellite's attitude dynamics, when the center of mass moves on a Keplerian trajectory. They found a stable relative equilibrium position in the case of circular orbits which disappears as far as elliptic trajectories are considered. They replaced the equilibrium position by planar periodic motions and they proved this motion is unstable with respect to out-of-plane perturbations. They also gave some numerical evidences of the existence of stable spatial periodic motions.

Burov et al. [5] considered the motion of a dumbbell–shaped body in an attractive Newtonian central field. They used the Poincare’s theory to determine the conditions for the existence of families of system periodic motions depending on the arising small parameter and passing into some stable radial steady–state motion of the unperturbed problem as the small parameter tends to zero. They also proved that, each of the radial relative equilibria generates one family of such periodic motions, for sufficiently small parameter values. Furthermore, they studied the stability of the obtained periodic solutions in the linear approximation as well as these solutions are calculated up to terms of the first order in the small parameter.

Guirao et al. (2013) gave sufficient conditions for the existence of periodic solutions of the perturbed attitude dynamics of a rigid dumbbell satellite in a circular orbit.

The statement of our main results is the following.

**Theorem 1.** Let consider a dumbbell satellite moving in a gravity field generated by an oblate body considering the effect of the zonal harmonic parameter $A$. The pass trajectory of the mass center of the system is periodic and different from the classical one. If $A$ is equal to zero our solution coincides with the elliptical classical one.

Finally, considering the motion in the satellite approximation we complete the classical theory, stating the following result.

**Theorem 2.** The equations of motion in the satellite approximation can be reduced to Beletsky’s equation when $A$ is equal to zero.
Note that Theorem 1 generalizes Celletti and Sidorenko [6], Burov and Dugain [4] and Nakanishi et al. [15] due to oblateness parameter.

The structure of the paper is as follows. In Section 2 we present in details the model description, presenting the deduction of the potential, the kinetic energy and the Lagrangian function of the system. In Section 3 we present the morphology of the equations of motion and the equation of the mass center of the system. In Sections 4 and 5 we respectively provide the prove of Theorems 1 and 2.

2. Model description

2.1. Hypothesis. We assume that the dumbbell satellite is formed by massless rod of length \( l \) with to masses \( m_1 \) and \( m_2 \) placed at its ends. Let \( c \) be the center of two masses moving in a gravity field generated by an oblate body whose mass \( m \) is located at 0 where the distance between 0 and \( c \) is \( r \) and \( r \gg l \).

Let us consider the orbital reference frame \( cxy \) with origin at the dumbbell’s center, and the polar coordinates of the center are \( (r, \theta) \). While the rotation of the satellite relative to ray \( oc \) will be determined by an angle \( \Theta \). Furthermore we denote the reduced mass by \( \mu \) and the sum of the two masses by \( m_s \) where \( \mu = m_1 m_2 / m_s \) and \( m_s = m_1 + m_2 \), see Figure 1 for details.

Now, we assume that \( \vec{r}_i \) is the position vector of \( m_i \) with respect to 0. Moreover, let the vector \( \vec{n}_i \) denotes the position vector of \( m_i \) with respect to the center of mass of the dumbbell satellite, \( i \in \{1, 2\} \).

Therefore, the magnitudes of the position vectors \( \vec{r}_i \) are controlled by
(1) \[ r_i^2 = r^2 + n_i^2 + 2(-1)^{2-i} n_i r \cos \Theta \]

where

(2) \[ n_i = \frac{m_{3-i} l}{m_s} \]

2.2. The potential of the model. From the potential theory, the gravitational potential (any object has axial symmetry \( m_0 \)) experienced by the satellite \( m \) up to \( J_2 \) will be controlled by (see Murray and Dermott [8]).

\[ V_0 = -Gm_0 m \left( \frac{1}{r_0} + \frac{J_2 R^2}{2r_0^3} \right) \]

Note that here \( G \) is the gravitational constant, \( m \) is the mass of the satellite and \( m_0 \) is the mass of an oblate object while \( r_0 \) is the distance between \( m \) and \( m_0 \) as well as \( R \) and \( J_2 \) are the main radius and zonal harmonic coefficient of the oblate body respectively.

If we assume that \( R \) represent the unit of distance , \( m_0 \) is also the unit mass and denote \( J_2 \) by \( A \). We have that the potential experienced by the masses \( m_1 \) and \( m_2 \) are \( V_1 \) and \( V_2 \) such that

(3) \[ V_1 = -Gm_1 \left( \frac{1}{r_1} + \frac{A}{2r_1^3} \right) \]

(4) \[ V_2 = -Gm_2 \left( \frac{1}{r_2} + \frac{A}{2r_2^3} \right) \]

Therefore the total potential \( V \) can be written as

(5) \[ V = -k \left( \frac{m_1}{r_1} + \frac{m_2}{r_2} + A \left( \frac{m_1}{2r_1^3} + \frac{m_2}{2r_2^3} \right) \right) \]

where \( k = G \) denotes the gravity parameter associated to the oblate body.

2.3. The kinetic energy of the model. To construct the kinetic energy of our system, let the vectors \( \xi_1 \) and \( \xi_2 \) be an orthogonal set of unitary vectors with \( \xi_1 \) corresponding to the direction from 0 to \( c \).

Consider \( i \) and \( j \) be another orthogonal set of unitary vectors such that \( \vec{i} \) is a vector in the direction of \( x \) axis. Consequently the vectors of the locations \( \xi_i \) and associates velocities \( \xi_i \) of masses \( m_i \) can be written as

\[ \xi_i = \vec{r} + n_i \]

\[ \xi_i = \frac{d\vec{r}_i}{dt} \]

where
\[ r = r (\cos \theta i + \sin \theta j) \]
\[ n_i = (-1)^i n_i (\cos \Theta e_1 + \sin \Theta e_2) \]
\[ e_i = (-1)^i (\cos (\theta + \frac{\pi}{4}) \hat{i} + \sin (\theta + \frac{\pi}{4}) \hat{j}) \]

Therefore, after some calculations, we obtain

\[ v_i^2 = \begin{cases} \dot{r}^2 + r^2 \ddot{\theta}^2 + n_i^2 (\dot{\theta} + \dot{\Theta})^2 \\ -2(-1)^i r n_i (\dot{\theta} + \dot{\Theta}) \sin \Theta \\ +2(-1)^i r n_i (\dot{\theta} + \dot{\Theta}) \cos \Theta \end{cases} \]

Since the kinetic energy of the dumbbell satellite system is

\[ T = \frac{1}{2} \sum_{i=1}^{2} m_i v_i^2 \]

Substituting equation (6) into (7), the kinetic energy can be written in the form

\[ T = T_s + T_r \]

where

\[ T_s = \frac{1}{2} m_s (\dot{r}^2 + r^2 \dot{\theta}^2) \]
\[ T_r = \frac{1}{2} \mu l^2 (\dot{\theta} + \dot{\Theta})^2 \]

Hence

\[ T = \frac{1}{2} m_s (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} \mu l^2 (\dot{\theta} + \dot{\Theta})^2 \]

2.4. The Lagrangian function of the model. Since the Lagrange’s function is defined by \( L = T - V \) from equations (3), (4) and (9) we get

\[ L = \frac{1}{2} m_s (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} \mu l^2 (\dot{\theta} + \dot{\Theta})^2 + k \left( \frac{m_1}{r_1} + \frac{m_2}{r_2} + A \left( \frac{m_1}{2r_1^3} + \frac{m_2}{2r_2^3} \right) \right) \]

Therefore, the equations of motion will be governed by

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\chi}} \right) - \frac{\partial L}{\partial \chi} = 0, \ \chi \in \{r, \theta, \Theta\} \]
3. Equation of motion

3.1. Equations of motion for the general case. Substituting equation (10) into (11) when \( \chi \in \{ r, \theta, \Theta \} \) the equations of motion can be written in the following form

\[
(m_s r^2 + \mu l^2) \ddot{\theta} + \mu l^2 \ddot{\Theta} = p_\theta = Q
\]

where \( Q \) constant and

\[
m_s (\ddot{r} - r \left( \frac{p_\theta - \mu l^2 \ddot{\Theta}}{m_s r^2 + \mu l^2} \right)^2) = -k \left\{ \frac{m_1 (r + n \cos \Theta)}{r_1^4} + \frac{m_2 (r - (l-n) \cos \Theta)}{r_2^4} + \frac{3}{2} \frac{A}{r_1^5} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \right\}
\]

\[
\ddot{\Theta} + \frac{2 \dot{r} (\mu l^2 \dot{\Theta} - p_\theta)}{r (m_s r^2 + \mu l^2)} = -k (m_s r^2 + \mu l^2) \sin \Theta \left( \frac{1}{r_1^3} - \frac{1}{r_2^3} \right) + \frac{3}{2} \frac{A}{r_1^5} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)
\]

Where \( n_1 = n \) and \( n_2 = l - n \) while \( p_\theta \) is a constant expresses the angular momentum conservation.

3.2. Dumbbell’s center of motion. Since \((r, \theta)\) is the coordinate of the dumbbell’s center, therefore the kinetic energy \( T_s \) is given by equation (8), while the potential of the center of mass \( V_s \) is given by

\[
V_s = -G m_s \left( \frac{1}{r} + \frac{A}{2r^3} \right)
\]

Consequently, the Lagrange function \( L_s \) of the center of mass can be represented in the form

\[
L_s = \frac{1}{2} m_s (\dot{r}^2 + r^2 \dot{\theta}^2) + G m_s \left( \frac{1}{r} + \frac{A}{2r^3} \right)
\]

Substituting equation (12) into (11) with \( L = L_s \) and \( \chi \in \{ r, \theta \} \) and taking account that the equations of motion can be written on the form

\[
\frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{r}} \right) - \frac{\partial L_s}{\partial r} = 0,
\]

\[
\frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{\theta}} \right) - \frac{\partial L_s}{\partial \theta} = 0,
\]

we state that the motion of dumbbell’s center will be controlled by

\[
\dot{\dot{r}} - r \dot{\theta}^2 = -k \left( \frac{1}{r^2} + \frac{3A}{2r^4} \right)
\]

\[
m_s r^2 \dot{\theta} = F \text{ or } \dot{\theta} = h
\]

where \( F \) is a constant and \( h \) is the angular momentum which is constant too, that can be evaluated by the initial conditions.

Let be \( r = \frac{1}{u} \), consequently
(13) \[ \frac{d^2u}{d\theta^2} + u = \frac{k}{h^2} + \frac{3}{2}kAu^2 \]
is the equation which represents the dumbbell’s center motion, but we can assume that this motion will be followed a Kepler’s orbit when the effect of oblateness parameter is absent ($A = 0$). Consequently the solution can be written as
\[ r_0 = \frac{h^2/k}{(1 + e_0 \cos \theta)} \]
where $r_0 = \frac{1}{u_0}$, $e_0$ is the orbit eccentricity such that $0 \leq e_0 < 1$, in the framework of elliptic orbits and $\theta$ is a true anomaly of the center of mass. When $\theta = 0$, $u_0 = \frac{1}{r_p} = \frac{k}{h^2}(1 + e_0)$, $r_p = a(1 - e_0)$ is the pericenter (periapsis) and $a$ is a semi–major axis.

4. Proof of Theorem 1

We look for solutions in the form $u(\theta, \epsilon)$ under the condition $0 < \epsilon \ll 1$. Since $A = J_2$ and $J_2 \in [1 \times 10^{-3}, 1 \times 10^{-6}]$ for the most of celestial bodies then we can replace $\epsilon$ by $A$. In addition, this solution must hold the initial conditions
\[ u(0, A) = \frac{1}{r_p}, \]
\[ D_\theta u(0, A) = 0. \]

Therefore, we search for straight forward expansion of an asymptotes solution as $A$ tends to zero in the following form

(14) \[ u(\theta, A) = u_0(\theta) + Au_1(\theta) + o(A^2). \]

Since we are considering the effect of the zonal harmonic of $J_2$, but the perturbation due to $J_2$ is of order about $10^{-3}$ of the unperturbed main term $(m_1/r_1)$ or $(m_2/r_2)$. While all other coefficients of zonal harmonic are about $10^{-6}$ or less. Therefore, it is sufficient from practical point of view, we take the expansion in equation (14) up to $A$. On the other hand, $o(A^2)$ represent the effect of the zonal harmonic $J_4$ while our potential does not contain the zonal harmonic $J_4$. Consequently we truncate the expansion in equation (14) up to the linear term $A$. In this case the leading–order perturbation equations are
\[ D_\theta u_0 + u_0 = \frac{k}{h^2}, \]
\[ D_\theta u_1 + u_1 = \frac{3}{2}k u_0^2. \]
Under the conditions $u_0(0) = \frac{1}{r_p}$, $D_\theta u_0 = 0$, $u_1(0) = 0$ and $D_\theta u_1 = 0$, hence the solution is governed by

$$u_0 = \frac{k}{h^2} (1 + e_0 \cos \theta)$$

and

$$u_1 = \frac{3k^3}{2h^4} [1 + \frac{1}{2} e_0^2 - \frac{1}{3} e_0^2 \cos \theta + e_0 \theta \sin \theta - \frac{1}{6} e_0^2 \cos 2\theta].$$

Therefore, the general expression of the dumbbell’s center motion up to $o(A)$ will be governed by

$$(15) \quad u(\theta, A) = u_0(\theta) + Au_1(\theta).$$

Equation (15) represents a solution which contains a secular term that grows in $\theta$. As a result, the expansion is not uniformly valid in $\theta$ and breaks down when $\theta = o(A)$, furthermore $Au_1$ is no longer a small correction of $u_0$. But convergent series approximation of the periodic solution can be determined by the continuation method known as the Lindstedt–Poincare’s technique.

Since equation (13) is a second order differential equation, it describes a dynamical system in which A is a small parameter. Consequently if $A = 0$ the system will be reduced to a harmonic oscillator which has a solution with period $T = 2\pi/\omega_0$ where $\omega_0 = 1$.

The continuation method enables us to construct a periodic solution for $A \neq 0$. If we consider that the angular velocity changes due to the non–linear terms, the asymptotic solution $u(\theta, A)$ and the angular velocity $\omega$ of the dynamical system can be expanded as

$$(16) \quad u(\theta, A) = u_0(\theta) + Au_1(\theta) + A^2 u_2(\theta) + \ldots$$

$$\omega = 1 + A\omega_1 + A^2 \omega_2 + \ldots$$

To construct a uniformly valid solution, we will introduce a stretched variable $\tau = \omega \theta$, therefore

$$(17) \quad \frac{d}{d\theta} = \omega \frac{d}{d\tau}, \quad \frac{d^2}{d\theta^2} = \omega^2 \frac{d^2}{d\tau^2}.$$ 

Substituting equations (17) into (13) we obtain

$$(18) \quad \omega^2 \frac{d^2 u}{d\tau^2} + u = \frac{k}{h^2} + \frac{3}{2} k Au^2$$

Now, under the following conditions

$$u(0, A) = \frac{1}{r_p},$$

$$u(0, A) = 0,$$
we insert the series expansion (16) into (18) and equating terms of the same order in \( A \) with keeping the terms up to first order of \( A \), we obtain the following:

- The coefficient of \( A^0 \) gives a homogeneous equation in the form

\[
\frac{d^2u_0}{d\tau^2} + u_0 = \frac{k}{h^2}
\]

where \( u_0(0) = \frac{1}{r_p}, \frac{du_0(0)}{d\tau} = 0 \) and \( u_0(\tau + 2\pi, A) = u_0(\tau, A) \) with a solution

\[
u_{0g}(\tau) = \frac{k}{h^2} (1 + \epsilon_0 \cos \tau)
\]

- The coefficient of \( A \) gives a non–homogeneous equation in the form

\[
\frac{d^2u_1}{d\tau^2} + u_1 = a_1 + a_2 \cos \tau + a_3 \cos 2\tau
\]

where \( u_1(0) = 0, \frac{du_1(0)}{d\tau} = 0 \) and

\[
a_1 = \frac{3k^3}{2h^4} (1 + \frac{1}{2} \epsilon_0^2),
\]

\[
a_2 = \frac{3\epsilon_0 k^3}{h^4} (1 + \frac{2\omega_1 h^2}{3k^2}),
\]

\[
a_3 = \frac{3\epsilon_0^2 k^3}{4h^4}
\]

with a particular solution

\[
u_1(\tau + 2\pi, A) = u_0(\tau, A)
\]

\[
u_{1p} = a_1 + \frac{1}{2} a_2 \cos \tau + \frac{1}{2} a_2 \tau \sin \tau - \frac{1}{3} a_3 \cos 2\tau.
\]

This solution contain a secular term \( a_2 \tau \sin \tau/2 \), to avoid this term and the solution becomes periodic we have to equate its coefficient by zero, hence

\[
\omega_1 = -\frac{3k^2}{2h^2}.
\]

Therefore the general solution of equation (20) is controlled by

\[
u_{1g} = \frac{3k^3}{2h^4} (1 + \frac{1}{2} \epsilon_0^2) - \frac{k^3}{2h^2} (3 + \epsilon_0^2) \cos \tau - \frac{\epsilon_0^2 k^3}{4h^4} \cos 2\tau.
\]

Substituting equations (19) and (21) into (16), the general solution of equation (18) becomes

\[
u = \frac{k}{h^2} (1 + k_1)[1 + (\frac{\epsilon_0 - k_2}{1 + k_1}) \cos \tau + \frac{k_3}{1 + k_1} \cos 2\tau]
\]
where

\[ k_1 = \frac{3}{4}Ak^2(2 + e_0^2) \]
\[ k_2 = \frac{1}{2}Ak^2(3 + e_0^2) \]
\[ k_3 = \frac{Ak^2e_0^2}{4h^2} \]
\[ \tau = (1 - \frac{3Ak^2}{2h^2})\theta \]

Therefore

\[ r = \frac{h^2/\bar{k}}{(1 + e \cos \tau + \bar{e} \cos 2\tau)} \]

with

\[ \bar{k} = k(1 + k_1) \]
\[ e = \left( e_0 - k_2 \right) \]
\[ \bar{e} = \frac{k_3}{1 + k_1} \]

In short, it is clear that the trajectory of the mass center differs from that obtained by Celletti and Sidorenko [6], Burov and Dugain [4] and Nakanishi et al. [15] due to oblateness parameter. Although, this solution is periodic. While this trajectory is the same as their solutions when the effect of oblateness is ignored. Since \( e_0 < 1 \) and \( A << 1 \) as a result \( Ae_0^2 << 1 \) is very small. Therefore, if we neglect all terms that include \( Ae_0^2 \), the equation (22) will be reduced to

\[ r = \frac{h^2/\bar{k}}{(1 + e \cos \tau)}, \]
\[ \bar{k} = k(1 + \frac{3}{2}Ak^2), \]
\[ e = e_0 - \frac{3}{2}Ak^2(1 + e_0). \]

This means that the trajectory of the mass center is elliptic as the classical case with the decreasing of the elliptical parameter and the eccentricity, ending the proof.
5. Proof of Theorem 2

We shall start by the deduction of the equations of motion in satellite approximation. Indeed, substituting equations (1) and (2) into (5), the approximation of the potential energy can be written as

\[ V = -k\left(\frac{m_s}{r} + \frac{A}{2r^3}\right) + \frac{\mu l^2}{2r^3}(3\cos^2\Theta - 1). \]

In this potential we neglect all terms that contain coefficients \((1/r)\) with power four or more, since \(l \ll r\). Therefore the Lagrangian function becomes

\[ L = \frac{1}{2}m_s(\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2}\mu l^2(\dot{\theta} + \dot{\Theta})^2 + k\left(\frac{m_s}{r} + \frac{A}{2r^3}\right) + \frac{\mu l^2}{2r^3}(3\cos^2\Theta - 1). \]  

Substituting equation (23) into (11), the approximation equations of motion are

\[ m_s(\ddot{r} - r\dot{\theta}^2) = -k\left(\frac{m_s}{r^2} + \frac{3A}{2r^4}\right) + \frac{3\mu l^2}{2r^4}(3\cos^2\Theta - 1) \]
\[ (m_s r^2 + \mu l^2)\dot{\theta} + \mu l^2\dot{\Theta} = p_\theta \]
\[ \mu l^2(\dot{\theta} + \dot{\Theta}) = -\frac{3k\mu l^2}{r^3} \cos \Theta \sin \Theta \]

Now replacing the independent variable \(t\) with the starched variable \(\tau\) where \(\tau = \omega \theta\) and \(r^2 \dot{\theta} = h\) therefore, it is possible to write \(\dot{\tau} = \Omega(\tau)\) such that

\[ \Omega(\tau) = \frac{\omega k^2}{h^3}(1 + e \cos \tau + \overline{e} \cos 2\tau)^2. \]

Hence

\[ \frac{d}{dt} = \Omega \frac{d}{d\tau} \quad \frac{d^2}{dt^2} = \Omega^2 \frac{d^2}{d\tau^2} + \Omega \Omega' \frac{d}{d\tau} \]

where \((\cdot)\)' means \(\frac{d}{d\tau}\).

Inserting equations (26) into (24) and using equation (25), we obtain

\[ (1 + e \cos \tau + \overline{e} \cos 2\tau)\ddot{\Theta} - 2(e \sin \tau + 2\overline{e} \sin 2\tau)(\frac{1}{\omega} + \dot{\Theta}) + \frac{3k}{\omega^2 k} \cos \Theta \sin \Theta = 0. \]

Since \(\dot{\Theta} = d\Theta/d\tau\) and \(\tau = \omega \theta\), we can rewrite equation (27) in the form...
\begin{equation}
(1 + e \cos \omega \theta + \bar{e} \cos 2\omega \theta) \frac{d^2 \Theta}{d\theta^2} - 2\omega (e \sin \omega \theta + 2\bar{e} \sin 2\omega \theta)(1 + \frac{d\Theta}{d\theta})
+ \frac{3k}{k} \cos \Theta \sin \Theta = 0
\end{equation}

where

\[ \omega = 1 - \frac{3Ak^2}{2h^2}, \]
\[ \bar{e} = \frac{Ak^2 e_0^2}{4h^2}, \]

and

\[ \bar{k} = k[1 + \frac{3}{4}Ak^2(2 + e_0^2)]. \]

Now, for finishing only remark that equation (28) can be reduced to Beletsky’s equation, see [1] for more details, if we assume the oblateness effect is not consider (i.e., \( A = 0 \)) obtaining the relation

\[ (1 + e \cos \theta) \frac{d^2 \Theta}{d\theta^2} - 2e \sin \theta \frac{d\Theta}{d\theta} + 3 \cos \Theta \sin \Theta = 2e \sin \theta, \]

which ends the proof.

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