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The existence of the extremal solution for the boundary value problems of variable fractional order differential equation with causal operator

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ABSTRACT
In the present study, the two-point boundary value problem is considered for the variable fractional order differential equation with causal operator. Under the definition of the Caputo-type variable fractional order operators, the necessary inequality and the existence results of the solution is obtained for the variable order fractional linear differential equations according to Arzela-Ascoli theorem. Then, based on the proposed existence results and the monotone iterative technique, the existence of the extremal solution is studied, and the relative results are obtained based on the lower and upper solution. Finally, an example is provided to illustrate the validity of the theoretical results.

KEYWORDS
Two-point boundary value problem; Caputo-type variable fractional order operators; Arzela-Ascoli theorem; Monotone iterative technique; The extremal solution

1. Introduction

It's well known that constant order fractional calculus has been viewed as a basic tool to describe the natural phenomenon of the practical problems in recent years [1–5]. However, there exists many complex behavior in engineer practice which can not be described by the constant order fractional order models. Thus, the variable order fractional operator is proposed to model the complex phenomena.

The variable order (VO) fractional operator is originated at the end of the 20th century. In 1998, Lorenzo and Hartley propose the concept of VO fractional operator [6]. In 2002, the concept is discussed more deeply, which connects with the physical process [7]. Then, the VO fractional operator is successful applied to model the practical problems of the engineering. And it arouses extensive attention between researchers.

Recent years, the theory of VO fractional derivatives and integrals is obtained a

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wide of study, but it's still in the infancy at present. At the turn of the century, it is used to simulate the temporal or spatial correlation phenomena [8,9]. According to the concept which is put forward by Lorenzo and Hartley [6], the VO fractional operator is treated as a generalization of the constant order fractional operator [10, 11]. So far, there exists several definitions of VO fractional derivative which includes Riemann-Liouvile, Caputo, Marchaud [12], and Coimbra type fractional operator definition [13], each of which have a special meaning to meet the desired objectives [14, 15]. With the deepening of research, a series of research hotspots are emerging such as the existence, uniqueness of solutions of differential operators with variable fractional order.

In the past several decades, there exists several results on the existence of the solution for constant order fractional differential equation [16–25]. Moreover, the boundary value problem has become an important research topic in the area of constant order fractional differential equations. For example, Jiang et al [26] studied the two-point boundary value problems for fractional differential equation with causal operator by lower and upper solution method and the monotone iterative technique. Shuqin Zhang [27] studied the nonlinear boundary value problem of a VO fractional differential equation, and obtained the existence result. Besides, Zhang [28] obtained the existence results of solutions for the boundary value problem based on the Schauder fixed point theorem. However, there few results on the boundary value problem of a VO differential equation with causal operator by monotone iterative technique. Motivated by this, we apply the monotone iterative technique to investigate boundary value problem of a VO fractional differential equation.

According to the definitions of the VO fractional operator, it's well known that the VO fractional operator is more complex than the constant order fractional operator. Because it's kernel is related to a variable exponent. Moreover, the VO fractional operator is irreversible, and thus the VO fractional differential equations can not be simply transformed into the equivalent Volterra integral equations. Thus, this requires that the research on the existence of solutions for VO fractional differential equations needs another methods.

The rest of paper is presented as follows: In section 2, some definitions and lemmas are listed. In section 3, The necessary inequality and the existence results of the solution are obtained for the VO fractional differential equation. Section 4 gives the theorem of the existence of extremal solution for the VO fractional differential equation. Section 5 illustrates an example to verify the the theoretical results.

In this paper, we consider the following system

$$\begin{cases} {}^{C}_{0}D^{q(t,s)}_{t}x(t) = (Qx)(t), t \in [0, T], \\ H(x(0), x(T)) = 0. \end{cases} \tag{1}$$

where $0 < q_1 \le q(t,s) \le q_2 < 1$.

2. Preliminaries

There exist many kinds of definitions for VO fractional derivative and integral [7], [14] and [29]. In this paper, the Caputo type definition is adopted due to its extensive application in engineering fields.

Definition 2.1. [7] The Caputo type definition of VO fractional integration is defined

as follows:

$$I_t^{q(t,s)}x(t) = \frac{1}{\Gamma(q(t))} \int_0^t (t-s)^{q(t)-1}x(s)ds, 0 < q(t,s) < 1, \tag{2}$$

where, $\Gamma(\cdot)$ is the Gamma function, $t \in [0,T]$.

$$I_t^{q(t,s)}x(t) = \int_0^t \frac{(t-s)^{q(t-s)-1}}{\Gamma(q(t-s))}x(s)ds, 0 < q(t,s) < 1, \ t \in [0,T], \tag{3}$$

and

$$I_t^{q(t,s)}x(t) = \int_0^t \frac{(t-s)^{q(s)-1}}{\Gamma(q(s))}x(s)ds, 0 < q(t,s) < 1, \ t \in [0,T], \tag{4}$$

if the integration is point-wise defined.

Definition 2.2. [7] The Caputo type definition of VO fractional derivatives are defined if the integration is point-wise defined as follows:

$$_0^C D_t^{q(t,s)}x(t) = \frac{1}{\Gamma(1-q(t))} \int_0^t (t-s)^{-q(t)}x'(s)ds, 0 < q(t,s) < 1, \tag{5}$$

where, $t \in [0,T]$,

$$_0^C D_t^{q(t,s)}x(t) = \int_0^t \frac{(t-s)^{-q(t-s)}}{\Gamma(1-q(t-s))}x'(s)ds, 0 < q(t,s) < 1, \ t \in [0,T], \tag{6}$$

and

$$_0^C D_t^{q(t,s)}x(t) = \int_0^t \frac{(t-s)^{-q(s)}}{\Gamma(1-q(s))}x'(s)ds, 0 < q(t,s) < 1, \ t \in [0,T]. \tag{7}$$

Due to the complex of the VO fractional operators, the VO fractional calculus $D^{q(t,s)}$, $I^{q(t,s)}$ **do not** satisfy the following property generally

$$I^{p(t,s)}I^{q(t,s)} = I^{p(t,s)+q(t,s)}, \ D^{p(t,s)}I^{p(t,s)}f(t) = f(t). \tag{8}$$

Thus, it requires more technique to deal with the boundary problem of VO fractional system.

Definition 2.3. [30] The operator $Q : C([t_0,T),X) \to C([t_0,T),X)$ is said to be a causal operator if for each couple of elements $u,v \in C([t_0,T),X)$ such that $u(s) = v(s)$ for $t_0 \le s \le t$, the following equation holds

$$(Qu)(s) = (Qv)(s) \quad for \quad a.e. \ t_0 \le s \le t < T.$$

Definition 2.4. The solution of the boundary value problem of VO fractional system (1) is defined by a function $x \in C^1([0,T],\mathbb{R})$ which satisfies:
(i) $_0^C D_t^{q(t,s)}x(t) = (Qx)(t), t \in [0,T]$,
(ii) $H(x(0),x(T)) = 0$.

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Remark 1. If the VO parameter $q(t, s)$ is a constant, then, the above definition of the VO fractional operator is reduced to the definition of the usual constant order Caputo fractional order calculus.

For the fractional order differential equations, Arzela-Ascoli theorem is an important tool in discussing the existence of solutions. Thus, in this present study, it's applied to prove the existence of the extremal solution for the VO fractional differential equation.

Lemma 2.5. *([31])(Arzela-Ascoli) If a sequence $\{x_n(t)\} \in C[0,T]$ is uniformly bounded and equicontinuous, then it has a uniformly convergent subsequence.*

3. The existence results of the solution for variable order fractional linear differential equations

In this section, some necessary theorems that are used to prove the existence of the extremal solutions are provided in the following.

Theorem 3.1. *Assume $x \in C^1([0,T], \mathbb{R})$ satisfy:*

$$
\begin{cases}
{}^c D^{q(t,s)} x(t) \leq -(Lx)(t), & t \in [0, T], \\
x(0) \leq \delta x(T), & \delta \in [0, 1],
\end{cases}
\tag{9}
$$

for $0 < q_1 \leq q(t,s) \leq q_2 < 1$, when $q(t,s) = 1$, the above inequality is also satisfied. Simultaneously, the following inequalities are true

$$
\sup_{t \in [0,T]} \left\{ \frac{\bar{T}}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} (L\mathbf{1})(s) ds \right\} \leq 1,
\tag{10}
$$

where $L \in C(E, E)$, $E = C([0,T], \mathbb{R})$ is a positive linear operator. $\bar{T} = \frac{\Gamma(1-q_1)}{\Gamma(1-q_2)T^{q_2}\tilde{T}}$, $\mathbf{1}(t) = 1$, $t \in [0, T]$. Then $x(t) \leq 0$, $t \in [0, T]$.

Proof. Obviously, the following inequality is true

$$
T^{-q(t,s)} \leq \begin{cases}
(\frac{1}{T})^{q_2}, & 0 < T < 1, \\
1, & 1 \leq T < +\infty,
\end{cases}
\tag{11}
$$

with $\tilde{T} = \max\{T^{-q_2}, 1\}$, then, $T^{-q(t,s)} \leq \tilde{T}$.

In order to obtain the main results, the process is divided into the following situations.

Situation I. Assume $x(0) \leq 0$, and the inequality $x(t) \leq 0, t \in [0, T]$ is not true. Then, there exists $t^* \in [0, T]$ such that $x(t^*) > 0$. If $\delta = 0$, then $x(0) \leq 0$. Thus, $t^* \in (0, T]$. Set

$$
x(t_0) = \min_{[0, t^*]} x(t) \leq 0.
$$

Through a series of mathematical transformations, it follows that

$$
-(Lx)(t) \geq^c D^{q(t,s)} x(t)
$$

$$\geq \int_0^t \frac{(t-s)^{-q(t,s)}}{\Gamma(1-q(t,s))} x'(s)ds$$

$$\geq \frac{1}{\Gamma(1-q_1)} \int_0^t (\frac{t-s}{T})^{-q(t,s)} T^{-q(t,s)} x'(s)ds$$

$$\geq \frac{\tilde{T}}{\Gamma(1-q_1)} \int_0^t (\frac{t-s}{T})^{-q_2} x'(s)ds$$

$$\geq \frac{\tilde{T} T^{q_2}}{\Gamma(1-q_1)} \int_0^t (t-s)^{-q_2} x'(s)ds$$

$$\geq \frac{\tilde{T} T^{q_2} \Gamma(1-q_2)}{\Gamma(1-q_1)} {}^c D^{q_2} x(t).$$

Put $\bar{T} = \frac{\Gamma(1-q_1)}{\tilde{T} T^{q_2} \Gamma(1-q_2)}$, thus, we can get

$$
{}^c D^{q_2} x(t) \leq -\bar{T}(Lx)(t). \tag{12}
$$

Then, applying the fractional operator $I_{t_0+}^{q_2}$ to both sides of the differential inequality (12), it's obtained from the condition (10) that

$$I_{t_0+}^{q_2}({}^c D^{q_2} x(t)) \leq -\bar{T} I_{t_0+}^{q_2}(Lx)(t),$$

$$x(t^*) - x(t_0) \leq -\frac{\bar{T}}{\Gamma(q_2)} \int_{t_0}^{t^*} (t^* - s)^{q_2-1}(Lx)(s)ds$$

$$\leq -\frac{\bar{T} x(t_0)}{\Gamma(q_2)} \int_{t_0}^{t^*} (t^* - s)^{q_2-1}(L\mathbf{1})(s)ds$$

$$\leq -x(t_0),$$

then, $x(t^*) \leq 0$ which is a contradiction.

Situation II. Assume $x(0) > 0$. Then, $x(T) > 0$.

i) if $\delta = 1$.

Suppose $x(t) \geq 0$ on $[0, T]$ and $x(t) \not\equiv 0$. Then, $\int_0^T (T-s)^{q_2-1}(Lx)(s)ds > 0$.

Based on the boundary condition of the inequality (9), then, it results that by integrating the inequaty(12) from both sides

$$
\begin{cases}
x(0) \leq x(T), \\
x(T) \leq x(0) - \frac{\bar{T}}{\Gamma(q_2)} \int_0^T (T-s)^{q_2-1}(Lx)(s)ds,
\end{cases}
$$

which implies that the contradiction $\int_0^T (T-s)^{q_2-1}(Lx)(s)ds \leq 0$.

Suppose $x(t) < 0$, for $t \in (0, T]$. Set

$$x(t_1) = \min_{t \in [0,T]} x(t) = -\nu, \quad \nu > 0.$$

Based on the inequality (12), for $t \in [0, T]$, it's obtained that

$$
{}^c D^{q_2} x(t) \leq -\bar{T}(Lx)(t) \leq -\bar{T}(Lx)(t_1) \leq \nu \bar{T}(L\mathbf{1})(t).
$$

Integrating the above inequality from both sides, it implies that

$$x(T) \leq x(t_1) + \frac{\bar{T}\nu}{\Gamma(q_2)} \int_{t_1}^{T} (T-s)^{q_2-1}(L\mathbf{1})(s)ds$$

$$\leq -\nu + \frac{\bar{T}\nu}{\Gamma(q_2)} \int_{t_1}^{T} (T-s)^{q_2-1}(L\mathbf{1})(s)ds$$

$$\leq -\nu + \nu = 0.$$

Then $x(T) \leq 0$ which is a contradiction.

ii) if $0 < \delta < 1$.

Suppose $x(t) \geq 0$, for $t \in [0, T]$ and $x(t) \not\equiv 0$.

Integrating the inequality (12) from 0 to T, then, we can get

$$x(T) \leq x(0) - \frac{\bar{T}}{\Gamma(q_2)} \int_{0}^{T} (T-s)^{q_2-1}(Lx)(s)ds,$$

combined with the boundary conditions, then,

$$\begin{cases} x(0) \leq \delta x(T), \\ x(T) \leq x(0) - \frac{\bar{T}}{\Gamma(q_2)} \int_{0}^{T} (T-s)^{q_2-1}(Lx)(s)ds, \end{cases}$$

whcih implies

$$x(0) \leq -\frac{\delta}{1-\delta} \frac{\bar{T}}{\Gamma(q_2)} \int_{0}^{T} (T-s)^{q_2-1}(Lx)(s)ds.$$

Thus, integrating the inequality (12) can product the following inequalities

$$x(t) \leq x(0) - \frac{\bar{T}}{\Gamma(q_2)} \int_{0}^{t} (t-s)^{q_2-1}(Lx)(s)ds$$

$$\leq -\frac{r}{1-r} \frac{\bar{T}}{\Gamma(q_2)} \int_{0}^{T} (T-s)^{q_2-1}(Lx)(s)ds - \frac{\bar{T}}{\Gamma(q_2)} \int_{0}^{t} (t-s)^{q_2-1}(Lx)(s)ds$$

$$\leq 0.$$

Together with the assumption $x(t) \geq 0$, $t \in [0, T]$, it's derived that $x(t) \equiv 0$ which is a contradiction.

Suppose $x(t) < 0$. Set

$$x(t_2) = \min_{t \in [0,T]} x(t) = -\varepsilon, \quad \varepsilon > 0.$$

Based on the inequality (12), it's obtained that

$$^{c}D^{q_2}x(t) \leq -\bar{T}(Lx)(t) \leq \bar{T}\varepsilon(L\mathbf{1})(t).$$

and then,

$$x(T) \leq x(t_2) - \frac{\bar{T}}{\Gamma(q_2)} \int_{t_2}^{T} (t-s)^{q_2-1}(Lx)(s)ds$$

$$\leq -\varepsilon + \frac{\bar{T}\varepsilon}{\Gamma(q_2)} \int_{t_2}^{T} (t-s)^{q_2-1}(L\mathbf{1})(s)ds$$

$$\leq 0.$$

which is a contradiction. Thus, the proof is completed. □

Based on Theorem 3.1, the following general inequality can be got.

Theorem 3.2. *Assume $a(t) \in C([0,T], \mathbb{R})$ and $x \in C^1([0,T], \mathbb{R})$ satisfy the following inequality:*

$$\begin{cases} {}^cD^{q(t,s)}x(t) \leq -a(t)x(t) - (Lx)(t), & t \in [0,T], \\ x(0) \leq \delta x(T), & \delta \in [0,1] \end{cases} \tag{13}$$

for $0 < q_1 \leq q(t,s) \leq q_2 < 1$ with $0 \leq \delta\widehat{q}(T) \leq 1$, $\widehat{q}(t) = e^{-\bar{T}\int_0^t a(s)ds}$. When $q(t,s) = 1$, the above inequality (13) is also satisfied. And, combined with the following condition

$$\sup_{t \in [0,T]} \left\{ \frac{\bar{T}}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} I_{0+}^{1-q_2} [e^{\bar{T}\int_0^s a(\tau)d\tau}(L\widehat{q})](s)ds \right\} \leq 1. \tag{14}$$

Then $x(t) \leq 0, \quad t \in [0,T]$.

Proof. According to the inequality (13), we have

$$^cD^{q_2}x(t) \leq -\bar{T}a(t)x(t) - \bar{T}(Lx)(t). \tag{15}$$

with $\bar{T} = \frac{\Gamma(1-q_1)}{\tilde{T}T^{q_2}\Gamma(1-q_2)}$.

Put $m(t) = e^{\bar{T}\int_0^t a(s)ds}x(t)$, then, $m(t)$ satisfies the following equations:

$$\begin{aligned} {}^cD^{q_2}m(t) &= I_{0+}^{1-q_2}m'(t) \\ &= I_{0+}^{1-q_2}(\bar{T}e^{\bar{T}\int_0^t a(s)ds}a(t)x(t) + e^{\bar{T}\int_0^t a(s)ds}x'(t)) \\ &= I_{0+}^{1-q_2}e^{\bar{T}\int_0^t a(s)ds}[\bar{T}a(t)x(t) + x'(t)] \\ &\leq -\bar{T}I_{0+}^{1-q_2}[e^{\bar{T}\int_0^t a(s)ds}(Lx)(t)] \\ &\leq -\bar{T}I_{0+}^{1-q_2}[e^{\bar{T}\int_0^t a(s)ds}(LQ)(t)] \end{aligned}$$

with $Q(t) = \widehat{q}(t)m(t)$.

Then, by simple calculating, the system (13) is simplified to the form:

$$\begin{cases} {}^cD^{q_2}m(t) \leq -\bar{T}I_{0+}^{1-q_2}[e^{\bar{T}\int_0^t a(s)ds}(LQ)(t)], & for \ t \in [0,T], \\ m(0) \leq \delta_1 m(T), & for \ \delta_1 = \delta\widehat{q}(T). \end{cases}$$

Based on Theorem 3.1, the proof is completed. □

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Theorem 3.2 provides a basic tool for the following lemma according to Arzela-Ascoli theorem.

Theorem 3.3. *Consider the linear boundary problem*

$$\begin{cases} {}^cD^{q(t,s)}u(t) = -a(t)u(t) - (Lu)(t) + \theta(t), & t \in [0,T], \\ u(0) = \delta u(T) + \gamma, & \gamma \in \mathbb{R}, \end{cases} \tag{16}$$

for $u(t) \in C^1([0,T], \mathbb{R})$ and $0 < q_1 \leq q(t,s) \leq q_2 < 1$. When $q(t,s) = 1$, the above equality (16) is also satisfied. Let $a(t), \theta(t) \in C([0,T], R)$ and $L \in C(E,E)$ is a positive linear operator. Assume the condition (14) hold for $0 \leq \delta_1 < 1$ with $\delta_1 = \delta e^{-\bar{T} \int_0^T a(s)ds}$.
Then the linear problem (16) has a unique solution $u \in C^1([0,T], \mathbb{R})$.

Proof. We claim that there's only one solution at most for the linear problems (16).
In fact, a contradiction is made that it has two different solutions denoted by $\bar{x}_1, \bar{x}_2 \in C^1([0,T], \mathbb{R})$.
Set $X(t) = \bar{x}_1(t) - \bar{x}_2(t)$, then, we have the following problem:

$$\begin{cases} {}^cD^{q(t,s)}X(t) = -a(t)X(t) - (LX)(t), \\ X(0) = \delta X(T). \end{cases} \tag{17}$$

According to Theorem 3.2, it's obtained that $X(t) \leq 0$, then, $\bar{x}_1(t) \leq \bar{x}_2(t), t \in [0,T]$.
As the same way, put $X(t) = \bar{x}_2(t) - \bar{x}_1(T)$, then, we can get $X(t) \leq 0$. Thus, $\bar{x}_1(t) = \bar{x}_2(t)$, which implies that the linear problem has at most one solution.
The next step, we will prove that there exists at least one solution for the linear problem (16).
Set $v(t) = e^{\int_0^t a(s)ds}u(t)$, by the simple calculating, then,

$$\begin{aligned} {}^cD^{q(t,s)}v(t) &= {}^cD^{q(t,s)}[e^{\int_0^t a(s)ds}u(t)] \\ &= I_{0+}^{1-q(t,s)}[e^{\int_0^t a(s)ds}a(t)u(t) + e^{\int_0^t a(s)ds}u'(t)] \\ &= I_{0+}^{1-q(t,s)}[e^{\int_0^t a(s)ds}a(t)u(t) + e^{\int_0^t a(s)ds}(-a(t)u(t) - (Lu)(t) + \theta(t))] \\ &= -I_{0+}^{1-q(t,s)}[e^{\int_0^t a(s)ds}(Lu)(t)] + I_{0+}^{1-q(t,s)}(e^{\int_0^t a(s)ds}\theta)(t) \\ &= -(\bar{B}v)(t) + \theta^*(t), \end{aligned}$$

where,

$$\begin{cases} \bar{B} = I_{0+}^{1-q(t,s)}[e^{\int_0^t a(s)ds}(L\bar{v})](t), \\ \bar{v} = ve^{-\int_0^t a(s)ds}, \\ \theta^*(t) = I_{0+}^{1-q(t,s)}(e^{\int_0^t a(s)ds}\theta)(t). \end{cases}$$

Then the system (16) is transformed into the following form

$$\begin{cases} {}^cD^{q(t,s)}v(t) = -(\bar{B}v)(t) + \theta^*(t), & t \in [0,T], \\ v(0) = \delta_1 v(T) + \gamma, & \gamma \in \mathbb{R}, \end{cases} \tag{18}$$

where $\delta_1 = \delta e^{-\int_0^T a(s)ds}$.

Based on the property of the VO fractional operator, the following is to prove the existence result by the iteration method as following:

$$v_n(t) = v_{n-1}(t) + \int_0^t \frac{(t-s)^{-q(t,s)}}{\Gamma(1-q(t,s))} v_{n-1}(s)ds - Q(\int_0^t v_{n-1}(s)ds + v(0))(t), \quad (19)$$

where,

$$Q(\int_0^t v_{n-1}(s)ds + v(0))(t) = -(\overline{B}v)(t) + \theta^*(t).$$

Now, we will prove that the sequence $\{v_n(t)\}$ is uniformly bounded and equi-continuity.

Assume $\{v_{n-1}(t)\}$ is uniformly bounded on $[0,T]$ and let $|v_{n-1}(t)| \le \overline{G}_{n-1}$, $\overline{G}_f = \max_{0\le t\le T} |Q[t, \int_0^t v_{n-1}(s)ds + v(0)]|$.

Then, $v_n(t)$ satisfies the following inequalities

$$|v_n(t)| \le |v_{n-1}(t)| + \int_0^t |\frac{(t-s)^{-q(t,s)}}{\Gamma(1-q(t,s))} v_{n-1}(s)|ds + |Q[t, \int_0^t v_{n-1}(s)ds + v(0)]|$$

$$\le \overline{G}_{n-1} + \frac{\overline{G}_{n-1}}{\Gamma(1-q_1)} \int_0^t (t-s)^{-q(t,s)}ds + \overline{G}_f$$

$$\le \overline{G}_{n-1} + \frac{\overline{G}_{n-1}\widetilde{T}T^{q_2}}{\Gamma(1-q_1)} \int_0^t (t-s)^{-q_2}ds + \overline{G}_f$$

$$\le \overline{G}_{n-1} + \frac{\overline{G}_{n-1}\widetilde{T}T}{\Gamma(1-q_1)(1-q_2)} + \overline{G}_f < +\infty,$$

which implies that the sequence $\{v_n(t)\}$ is uniformly bounded.

For $0 \le t_1 \le t_2 \le T$, we have the following inequalities:

$$|v_n(t_2)-v_n(t_1)|$$

$$\le \left| \int_0^{t_2} \frac{(t_2-s)^{-q(t_2,s)}}{\Gamma(1-q(t_2,s))} v_{n-1}(s)ds - \int_0^{t_1} \frac{(t_1-s)^{-q(t_1,s)}}{\Gamma(1-q(t_1,s))} v_{n-1}(s)ds \right|$$

$$+ \left| Q(\int_0^{t_2} v_{n-1}(s)ds + v(0))(t_2) - Q(\int_0^{t_1} v_{n-1}(s)ds + v(0))(t_1) \right|$$

$$+ |v_{n-1}(t_2) - v_{n-1}(t_1)|$$

$$\le |v_{n-1}(t_2) - v_{n-1}(t_1)| + \left| \int_0^{t_1} \left[\frac{(t_2-s)^{-q(t_2,s))}}{\Gamma(1-q(t_2,s))} - \frac{(t_1-s)^{-q(t_1,s)}}{\Gamma(1-q(t_1,s))} \right] v_{n-1}(s)ds \right|$$

$$+ \left| Q(\int_0^{t_2} v_{n-1}(s)ds + v(0))(t_2) - Q(\int_0^{t_1} v_{n-1}(s)ds + v(0))(t_1) \right|$$

$$+ \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{-q(t_2,s)}}{\Gamma(1-q(t_2,s))} v_{n-1}(s)ds \right|$$

$$\le |v_{n-1}(t_2) - v_{n-1}(t_1)| + \frac{G_{n-1}}{\Gamma(1-q_1)} \int_0^{t_1} \left| (\frac{t_2-s}{T})^{-q(t_2,s)} - (\frac{t_1-s}{T})^{-q(t_2,s)} \right|.$$

$$T^{-q(t_2,s)}ds + G_{n-1} \int_0^{t_1} \left| \frac{((t_1-s)/T)^{-q(t_2,s)}}{\Gamma(1-q(t_2,s))} - \frac{((t_1-s)/T)^{-q(t_2,s)}}{\Gamma(1-q(t_1,s))} \right|.$$

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$$T^{-q(t_2,s)}ds + G_{n-1}\int_0^{t_1}\left|\frac{(t_1-s)^{-q(t_2,s)}}{\Gamma(1-q(t_1,s))} - \frac{(t_1-s)^{-q(t_1,s)}}{\Gamma(1-q(t_1,s))}\right|ds$$

$$+\left|Q(\int_0^{t_2}v_{n-1}(s)ds + v(0))(t_2) - Q(\int_0^{t_1}v_{n-1}(s)ds + v(0))(t_1)\right|$$

$$+\frac{\widetilde{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)}\left|\int_{t_1}^{t_2}(t_2-s)^{-q_2}ds\right|$$

$$\leq |v_{n-1}(t_2) - v_{n-1}(t_1)| + \frac{\widetilde{T}T^{q_1}G_{n-1}}{\Gamma(1-q_1)}\int_0^{t_1}\left|(t_2-s)^{-q_1} - (t_1-s)^{-q_1}\right|ds + \widetilde{T}T^{q_2}G_{n-1}\cdot$$

$$\int_0^{t_1}\left|\frac{(t_1-s)^{-q_2}}{\Gamma(1-q(t_2,s))} - \frac{(t_1-s)^{-q_2}}{\Gamma(1-q(t_1,s))}\right|ds + \frac{G_{n-1}}{\Gamma(1-q_1)}\left|\int_0^{t_1}\right|(t_1-s)^{-q(t_2,s)}$$

$$- (t_1-s)^{-q(t_1,s)}\Big|ds + \left|Q(\int_0^{t_2}v_{n-1}(s)ds + v(0))(t_2) - Q(\int_0^{t_1}v_{n-1}(s)ds + v(0))(t_1)\right|$$

$$+\frac{\widetilde{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)(1-q_2)}(t_2-t_1)^{1-q_2}$$

$$\leq |v_{n-1}(t_2) - v_{n-1}(t_1)| + \frac{\widetilde{T}T^{q_2}G_{n-1}}{\Gamma(2-q_1)}\left|t_1^{1-q_1} - t_2^{1-q_1} + (t_2-t_1)^{1-q_1}\right| + \widetilde{T}T^{q_2}G_{n-1}$$

$$\cdot\int_0^{t_1}\left|\frac{(t_1-s)^{-q_2}}{\Gamma(1-q(t_2,s))} - \frac{(t_1-s)^{-q_2}}{\Gamma(1-q(t_1,s))}\right|ds + \frac{G_{n-1}}{\Gamma(1-q_1)}\left|\int_0^{t_1}\left[(t_1-s)^{-q(t_2,s)}\right.\right.$$

$$\left.\left. - (t_1-s)^{-q(t_1,s)}\right]ds\right| + \frac{\widetilde{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)(1-q_2)}(t_2-t_1)^{1-q_2}$$

$$+\left|Q(\int_0^{t_2}v_{n-1}(s)ds + v(0))(t_2) - Q(\int_0^{t_1}v_{n-1}(s)ds + v(0))(t_1)\right|$$

Assume $\lim_{t_1\to t_2}|v_{n-1}(t_2) - v_{n-1}(t_1)| = 0$. Since the Gamma function $\Gamma(t)$ is continuous on $(0,1]$, then,

$$\frac{1}{\Gamma(1-q(t_1,s))} \to \frac{1}{\Gamma(1-q(t_2,s))} \quad \text{as} \quad t_1 \to t_2.$$

Besides, the continuity of the exponential function results in the following conclusion:

$$\lim_{t_1\to t_2}\int_0^{t_1}\left[(t_1-s)^{-q(t_2,s)} - (t_1-s)^{-q(t_1,s)}\right]ds = 0.$$

And,

$$\left|Q(\int_0^{t_2}v_{n-1}(s)ds + v(0))(t_2) - Q(\int_0^{t_1}v_{n-1}(s)ds + v(0))(t_1)\right|$$
$$\leq |(\overline{B}v)(t_2) - (\overline{B}v)(t_1)| + |\theta^*(t_2) - \theta^*(t_1)|$$

According to the condition of Theorem 3.3, the operators $\overline{B}$ and θ^* are bounded, then, $\lim_{t_1\to t_2}\left|Q(\int_0^{t_2}v_{n-1}(s)ds + v(0))(t_2) - Q(\int_0^{t_1}v_{n-1}(s)ds + v(0))(t_1)\right| = 0$.

Thus, we have $\lim_{t_1\to t_2}|v_n(t_2) - v_n(t_1)| = 0$, which implies that the set $\{v_n\}$ is equi-continuous. By Arzela-Ascoli theorem, there exists a converge subsequence which

converges uniformly to a continuous function v^* for $t \in [0, T]$. Taking the limit from the both side of the equation(19) when $n \to \infty$, then, we have that v^* is the solution of the system (18). The proof is completed. $\square$

4. The existence of the extremal solutions for the boundary problem

Definition 4.1. A function $x \in C^1([0, T], \mathbb{R})$ is said to be an upper solution of the boundary problem (1) if it satisfies:

$$\begin{cases} {}^cD^{q(t,s)}x(t) \le (Qx)(t), & t \in [0, T], \\ H(x(0), x(T)) \le 0. \end{cases}$$

Reversely, it is a lower solution of the boundary problem (1).

Definition 4.2. A function $u \in C^1([0, T], \mathbb{R})$ is called the maximal solution of problem (1) if $x(t) \le u(t), \ t \in [0, T]$, for each solution x of the boundary value problem (1). Reversely, it is a minimal solution.

In order to get the existence of the extremal solutions for the boundary problem (1), we give the following assumptions:

- (H_1): $(Qx)(t) \in C(E, E)$ and, $H \in C(R \times R, R)$, $E = C^1([0, T], R)$;

- (H_2): $u_0, \quad v_0 \in C^1([0, T], R)$ are lower and upper solutions of problem (1) respectively, and $u_0(t) \le v_0(t), \quad t \in [0, T]$;

- (H_3): the function $a \in C([0, T], R)$ satisfies

$$(Qx)(t) - (Qy)(t) \le a(t)[y(t) - x(t)] + L(y - x)(t),$$

  for $u_0(t) \le x(t) \le y(t) \le v_0(t), \quad t \in [0, T]$;

- (H_4): there exist constants $a, \ b$ satisfying $0 \le b \le a, \ a > 0$ and

$$H(\overline{x}, \overline{y}) - H(x, y) \le a(\overline{x} - x) - b(\overline{y} - y),$$

  for $u_0(0) \le x \le \overline{x} \le v_0(0)$ and $u_0(T) \le y \le \overline{y} \le v_0(T)$.

Theorem 4.3. *Assume that $(H_1), (H_2), (H_3), (H_4)$ and the condition (14) hold. Then, there exist two monotone sequences $\{u_n(t)\}, \ \{v_n(t)\}$ such that $\lim_{n \to \infty} u_n(t) = m(t)$, $\lim_{n \to \infty} v_n(t) = n(t)$, which $m(t), \ n(t)$ are the minimal and maximal solution of problem (1) respectively, with $u_0 \le m(t) \le n(t) \le v_0$.*

Proof. For the following linear boundary problem

$$\begin{cases} {}^cD^{q(t,s)}x(t) + a(t)x(t) = -(Lx)(t) + \delta_\zeta(t), \\ x(0) = \zeta(0) - \frac{1}{a}H(\zeta(0), \zeta(T)) + r[x(T) - \zeta(T)], \end{cases} \tag{20}$$

where $\delta_\zeta(t) = (Q\zeta)(t) + a(t)\zeta(t) + (L\zeta)(t), 0 \le r = \frac{b}{a} \le 1, \zeta \in C([0, T], R)$ and $u_0(t) \le \zeta(t) \le v_0(t)$.

Based on Theorem 3.3, the linear boundary problem (20) has a unique solution.

We claim that each solution $x(t)$ of the linear boundary problem (20) belongs to the set $[u_0(t), v_0(t)]$, $t \in [0, T]$, where $[u_0, v_0] = \{x \in C([0, T], \mathbb{R}) : u_0(t) \le x(t) \le v_0(t)\}$.

It's obtained from (H_2) that

$$^cD^{q(t,s)}u_0(t) \le -a(t)u_0(t) - (Lu_0)(t) + \delta_{u_0}(t).$$

Due to the condition (H_3), we can get that

$$(Qu_0)(t) - (Q\zeta)(t) \le a(t)[\zeta(t) - u_0] + L(\zeta - u_0)(t),$$

then, $\delta_\zeta(t) \ge \delta_{u_0}(t)$ which implies

$$^cD^{q(t,s)}x(t) \ge -a(t)x(t) - (Lx)(t) + \delta_{u_0}(t).$$

Thus,

$$
\begin{aligned}
^cD^{q(t,s)}(u_0 - x)(t) &=^c D^{q(t,s)}u_0(t) -^c D^{q(t,s)}x(t) \\
&\le -a(t)u_0(t) - (Lu_0)(t) + \zeta_{u_0}(t) + a(t)x(t) + (Lx)(t) - \zeta_{u_0}(t) \\
&= -a(t)(u_0 - x)(t) - L(u_0 - x)(t)
\end{aligned}
$$

and

$$(u_0 - x)(0) = u_0(0) - \zeta(0) + \frac{1}{a}H(\zeta(0), \zeta(T)) - r[x(T) - \zeta(T)].$$

According to the condition (H_4), it's obtained that

$$H(\zeta(0), \zeta(T)) - H(u_0(0), u_0(T)) \le a(\zeta(0) - u_0(0)) - b(\zeta(T) - u_0(T)).$$

Then,

$$
\begin{aligned}
(u_0 - x)(0) &\le u_0(0) - \zeta(0) + \frac{1}{a}H(\zeta(0), \zeta(T)) - r[x(T) - \zeta(T)] \\
&\le u_0(0) - \zeta(0) + \zeta(0) - u_0(0) - r\zeta(T) + ru_0(T) - r[x(T) - \zeta(T)] \\
&= r(u_0 - x)(T).
\end{aligned}
$$

Set $P(t) = u_0(t) - x(t)$, we have the following boundary problem:

$$
\begin{cases}
^cD^{q(t,s)}P(t) \le -a(t)P(t) - (LP)(t), \\
P(0) \le rP(T).
\end{cases}
\tag{21}
$$

Thus, in light of Theorem 3.2, we have the inequality $P(t) \le 0$, and then $u_0(t) \le x(t)$, $t \in [0, T]$. As the same method, it can be showed that $x(t) \le v_0(t)$, $t \in [0, T]$. Thus, $u_0(t) \le x(t) \le v_0(t)$.

In the following step, we construct two kinds of sequences $\{u_n\}, \{v_n\}$ which satisfy

the boundary value problem

$$\begin{cases} {}^{c}D^{q(t,s)}u_{n+1}(t) = (Qu_n)(t) - L(u_{n+1} - u_n)(t) - a(t)[u_{n+1}(t) - u_n(t)], \\ u_{n+1}(0) = u_n(0) - \frac{1}{a}H(u_n(0), u_n(T)) + r[u_{n+1}(T) - u_n(T)], \end{cases} \tag{22}$$

and

$$\begin{cases} {}^{c}D^{q(t,s)}v_{n+1}(t) = (Qv_n)(t) - L(v_{n+1} - v_n)(t) - a(t)[v_{n+1}(t) - v_n(t)], \\ v_{n+1}(0) = v_n(0) - \frac{1}{a}H(v_n(0), v_n(T)) + r[v_{n+1}(T) - v_n(T)]. \end{cases} \tag{23}$$

From the above results of the boundary problem(20), it's obtained that each of the boundary value problem (22) and (23) have a solution in the sector $[u_0(t), v_0(t)]$.

We claim that

$$u_0(t) \le u_1(t) \le u_2(t) \le \cdots \le u_n(t) \le v_n(t) \le \cdots \le v_2(t) \le v_1(t) \le v_0(t), \quad t \in [0, T].$$

In fact, the process is divided into the following cases based on the induction method;

(i): we show that $u_0 \le u_1$. Since u_0 is the lower solutions of problem (1), then, we have

$$\begin{cases} {}^{c}D^{q(t,s)}u_0(t) \le (Qu_0)(t), \\ {}^{c}D^{q(t,s)}u_1(t) = (Qu_0)(t) - (L(u_1 - u_0))(t) - a(t)[u_1(t) - u_0(t)]. \end{cases}$$

Put $P_1(t) = u_0(t) - u_1(t)$, then, $P_1(t)$ satisfies the following inequalities:

$$\begin{aligned} {}^{c}D^{q(t,s)}P_1(t) = {}^{c}D^{q(t,s)}u_0(t) - {}^{c}D^{q(t,s)}u_1(t) \\ \le (Qu_0)(t) - (Lu_0)(t) - (Qu_0)(t) + (Lu_1)(t) + a(t)[u_1(t) - u_0(t)] \\ = -a(t)P_1(t) - (LP_1)(t). \end{aligned}$$

The boundary condition is listed as

$$P_1(0) = u_0(0) - u_1(0) = \frac{1}{a}H(u_0(0), u_0(T)) - r[u_1(T) - u_0(T)] \le rP_1(T).$$

It's showed that $u_0 \le u_1$, for $t \in [0, T]$ according to Theorem 3.2.

(ii): Assume $u_{k-1}(t) \le u_k(t), \ t \in [0, T]$. By the induction hypothesis method, we claim $u_k(t) \le u_{k+1}(t), \ t \in [0, T]$.

Put $P_{k+1}(t) = u_k(t) - u_{k+1}(t)$, by simplifying the calculation, it's obtained from the condition (H_3) that,

$$\begin{cases} {}^{c}D^{q(t,s)}P_{k+1}(t) = -a(t)P_{k+1}(t) - (LP_{k+1})(t), \\ P_{k+1}(0) \le rP_{k+1}(T) \end{cases}$$

which is transformed by

$$\begin{aligned} {}^{c}D^{q(t,s)}P_{k+1}(t) = {}^{c}D^{q(t,s)}u_k(t) - {}^{c}D^{q(t,s)}u_{k+1}(t) \\ = (Qu_{k-1})(t) - (Qu_k)(t) - L(u_k - u_{k-1})(t) + L(u_{k+1} - u_k)(t) \\ - a(t)[u_k(t) - u_{k-1}(t)] + a(t)[u_{k+1}(t) - u_k(t)] \end{aligned}$$

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$$\leq a(t)[u_k(t) - u_{k-1}(t)] - L(u_k - u_{k+1})(t) - a(t)[u_k(t) - u_{k-1}(t)]$$
$$+ a(t)[u_{k+1}(t) - u_k(t)]$$
$$= -a(t)P_{k+1}(t) - (LP_{k+1})(t)$$

and

$$P_{k+1}(0) = u_k(0) - u_{k+1}(0)$$
$$= u_{k-1}(0) - \frac{1}{a}H(u_{k-1}(0), u_{k-1}(T)) + r[u_k(T) - u_{k-1}(T)] - u_k(0)$$
$$+ \frac{1}{a}H(u_k(0), u_k(T)) - r[u_{k+1}(T) - u_k(T)]$$
$$\leq u_{k-1}(0) - u_k(0) + \frac{1}{a}[a(u_k(0) - u_{k-1}(0)) - b(u_k(T) - u_{k-1}(T))]$$
$$+ r[u_k(T) - u_{k-1}(T)] - r[u_{k+1}(T) - u_k(T)]$$
$$= rP_{k+1}(T).$$

Then, the inequality $u_k(t) \leq u_{k+1}(t), \ t \in [0, T]$ is true. Thus, by the inductive hypothesis, it's obtained that

$$u_0(t) \leq u_1(t) \leq \cdots \leq u_k(t), \quad t \in [0, T].$$

By the same way, it can be showed that

$$v_k(t) \leq v_{k-1}(t) \leq \cdots \leq v_1(t) \leq v_0(t), \quad t \in [0, T].$$

(iii): The following is showed that $u_n(t) \leq v_n(t), \ t \in [0, T], \ n = 1, 2, \cdots$.
Set $\widehat{P}_n = u_n - v_n$. Similarly, the following inequality is true:

$$\begin{cases} {}^c D^{q(t,s)} \widehat{P}_n(t) = -a(t)\widehat{P}_n(t) - (L\widehat{P}_n)(t), \\ \widehat{P}_n(0) \leq r\widehat{P}_n(T) \end{cases}$$

which yields $u_n(t) \leq v_n(t), \ t \in [0, T], \ n = 1, 2, \cdots$ according to Theorem 3.2.
Thus,

$$u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \leq v_n(t) \leq \cdots \leq v_1(t) \leq v_0(t), \quad t \in [0, T].$$

Then, it implies from standard arguments that

$$\lim_{n \to \infty} u_n(t) = m(t) \text{ and } \lim_{n \to \infty} v_n(t) = n(t)$$

uniformly on $[0, T]$, where $m(t)$ and $n(t)$ are the solutions of the boundary problem (1).

Lastly, we show that $m(t)$ and $n(t)$ are the extremal solutions of the boundary problem (1).

Suppose $x(t)$ is any solution of the boundary problem (1) with $x(t) \in [z_0, y_0]$ and for some $k \geq 0, \ u_{k-1}(t) \leq x(t) \leq v_{k-1}(t), \ t \in [0, T]$.

Set $\bar{p}(t) = u_k(t) - x(t)$. From the condition (H_3), we obtain that

$$
\begin{aligned}
{}^{c}D^{q(t,s)}\bar{p}(t) &={}^{c} D^{q(t,s)}u_k -{}^{c} D^{q(t,s)}x(t) \\
&= (Qu_{k-1})(t) - L(u_k - u_{k-1})(t) - a(t)[u_k(t) - u_{k-1}(t)] - (Qx)(t) \\
&\leq a(t)[x(t) - u_{k-1}(t)] + L(x - u_k)(t) - a(t)[u_k(t) - u_{k-1}(t)] \\
&\leq -a(t)\bar{p}(t) - (L\bar{p})(t).
\end{aligned}
$$

and $\bar{p}(0) \leq r\bar{p}(T)$. Based on Theorem 3.2, it implies that $u_k(t) \leq x(t)$. Similarly, by the same process, it yields $x(t) \leq v_k(t)$.

Thus, by inductive hypothesis, it follows that $u_n(t) \leq x(t) \leq v_n(t), \quad for\ all\ n,\ t \in [0, T]$. Then, $m(t) \leq x(t) \leq n(t)$, which completes the proof. $\qquad\square$

5. An illustrative example

Consider the linear boundary problem

$$
\begin{cases}
{}^{c}D^{q(t,s)}x(t) = -a(t)x(t) + a(t)sin(x(t)) - t \int_0^t sx(s)ds, \\
0 = e^{x(0)} - x(1) - \frac{3}{4},
\end{cases}
\tag{24}
$$

where $a(t) \in C([0, 1], [0, \infty))$, and $(Qx)(t) = -a(t)x(t) + a(t)sin(x(t)) - t \int_0^t sx(s)ds$. Set $u_0(t) = 0, \quad v_0(t) = \delta$ with $1 \leq \delta \leq \frac{\pi}{2}$. Then

$$
(Qu_0)(t) = 0 ={}^{c} D^{q(t,s)}x(t),
$$

$$
H(u_0(0), u_0(1)) = -\frac{1}{4} \leq 0,
$$

$$
(Qv_0)(t) = -a(t)\delta + a(t)sin\delta - \delta\frac{t^3}{2} \leq -a(t)(\delta - sin\delta) \leq 0 ={}^{c} D^{q(t,s)}v_0(t),
$$

$$
H(v_0(0), v_0(1)) = e^\delta - \delta - \frac{3}{4} \geq 0,
$$

which means that $u_0,\ v_0$ are lower and upper solutions of problem (24), respectively. Besides,

$$
(Qx)(t) - (Qy)(t) = -a(t)[x(t) - y(t)] + a(t)[sin(x(t)) - sin(y(t))] - t \int_0^t s(x(s) - y(s))ds.
$$

$$
\leq a(t)[y(t) - x(t)] - t \int_0^t s(x(s) - y(s))ds
$$

for $0 \leq x(t) \leq y(t) \leq \delta$. Assume that

$$
\sup_{t \in [0,1]} \left\{ \frac{\bar{T}}{\Gamma(q_2)} \int_0^t (t - s)^{q_2-1} I_0^{1-q_2} [e^{\int_0^s a(\tau)d\tau} s \int_0^s \xi e^{-\int_0^\xi a(\tau)d\tau} d\xi](s)ds \right\} \leq 1.
$$

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Then, the extremal solutions of the boundary problem are existed (24).

6. Conclusion

The present paper is focused on the two-point boundary value problem of VO fractional differential equation with causal operator. The relative theorems about the necessary inequality and the existence results of the solution have been proposed. Based on the monotone iterative technique, the existence result of the extremal solution for VO fractional differential equation with causal operator has been obtained by the lower and upper solution. Lastly, an example has been listed to illustrate the validity of the theoretical results.

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