SHANNON–WHITTAKER–KOTEL'NIKOV'S THEOREM GENERALIZED

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ABSTRACT. Let $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be a bounded sequence of positive real numbers such that $\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| < \infty$ and $\sigma_\lambda(t) := \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)}$. The objective of this paper is to prove that σ_λ is an analytic function and for every $t \in \mathbb{R}$, $\sigma_\lambda(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} \operatorname{sinc}(t-k)\right)^n$. This result can be consider as the general sufficient condition type theorem which generalizes the Shannon–Whittaker–Kotel'nikov's recomposition theorem for the "exponential–trigonometric type maps" long time looked for application in the chemistry world. As a consequence we obtain, as an easy application, the results from [4] which were obtained by direct approach. Important applications to the signal theory are derived too since many signal can be written in σ_λ format for a proper sequence λ.

1. Introduction and statement of the main result

It is well–known that a central result in the signal theory is the Shannon–Whittaker–Kotel'nikov's recomposition theorem (see for instance [15], [21] or [24]) working for band–limited signals of $L^2(\mathbb{R})$ (i.e., for Paley–Wiener maps), and based on the normalized cardinal sinus map $\operatorname{sinc}(t)$ defined by

$$\operatorname{sinc}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0. \end{cases}$$

Another fundamental result on the signal processing theory is the Middleton's sampling theorem for band step functions (see [19]). This result was one of the first modifications of the classical Sampling Theorem which only works for band–limited maps, see [22]. After this starting point many different extensions and generalizations of this theorem have

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appeared in the literature trying to obtain approximations of non band–limited signals (see for instance [7] or [12]). Good surveys on these extensions are [8] or [24].

Recently, A. Antuña et al. [2, 3] inspired in the previous results, in the sense of trying to obtain approximations of non band–limited signals by using band–limited ones increasing the band size, proposed a generalization of the classical theorem.

But, the key point in this new approach which makes it completely different from the previous ones is that it keeps constant the sampling frequency generalizing in the limit the results of Marvasti et al. [17] and Agud et al. [1].

In this setting, [2, 3] states the following asymptotic property of type generalized sampling Shannon's theorem where the convergence of the series is considered in the Cauchy's principal value.

Property $\mathcal{P}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a map and $\tau \in \mathbb{R}^+$. We say that f satisfies the property $\mathcal{P}$ for τ if

$$f(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}} \left(\frac{k}{\tau} \right) \operatorname{sinc}(\tau t - k) \right)^n . \tag{1}$$

[2] proves that every constant signal holds property $\mathcal{P}$ for every given $\tau \in \mathbb{R}^+$ and conjectures that the Gaussian maps, i.e. maps of the form $e^{-\lambda t^2}$, $\lambda \in \mathbb{R}^+$ hold property $\mathcal{P}$ for every given $\tau \in \mathbb{R}^+$. To support the conjecture, [2] proves that the Gaussian map e^{-t^2} holds expression (1) for the three first coefficients of the power series representation of e^{-t^2}.

Moreover, the veracity of the conjecture is also suggested by the Boas's estimation [5]. Finally, [3] and [13] present proofs the conjecture using different approaches.

In [5] it is stated that if f has an integrable Fourier transform, the pointwise error between f and its sampling series $\sum f(k)\operatorname{sinc}(t-k)$ is controlled by $\int_{|\xi|>\frac{1}{2}} \left|\hat{f}(\xi)\right| d\xi$. Since $(e^{-\pi\lambda t^2})^{\frac{1}{n}} = e^{-\pi(t\sqrt{\frac{\lambda}{n}})^2}$ its Fourier transform is $\sqrt{\frac{n}{\lambda}} e^{-\pi(\xi\sqrt{\frac{n}{\lambda}})^2}$ and

$$\sqrt{\frac{n}{\lambda}} \int_{|\xi|>\frac{1}{2}} e^{-\pi(\xi\sqrt{\frac{n}{\lambda}})^2} d\xi = \int_{|\xi|>\sqrt{\frac{n}{4\lambda}}} e^{-\pi\xi^2} d\xi \to 0$$

as $n \to \infty$. Thus Boas's estimate proves that the integer samples of the n–th root of the Gaussian maps converge to the n–th root, in a sense consistent with (1).

[13] opened a door for applications to the chemical world because of the fact that pseudo–radioactive products which have a decomposition

dynamics determined by maps of the form $e^{-\lambda t^2}$ can be reconstructed using property $\mathcal{P}$. This application has been extended for other kind of maps like $e^{\cos(\pi t)}$ in M.T. de Bustos et al. [9] and $e^{\mathrm{sinc}\,(x-t)}$, $e^{\frac{\cos(\pi t)}{t}-\frac{\sin(\pi t)}{\pi t^2}}$ and $e^{\frac{\cos(\pi t)}{1+t^2}+\frac{\cosh\pi}{\sinh\pi}\frac{t\sin(\pi t)}{1+t^2}}$ in A. Antuña et al. [4].

The objective of the present paper is to prove property $\mathcal{P}$ for a general class of non–band limited maps. Moreover, as a consequence of our result, we obtain as corollaries the results of paper [4]. In short, some of the maps for which "ad hoc proves" have been presented in the literature till now for proving property $\mathcal{P}$ are individual members of our family. Therefore, we can say that our result is a general sufficient condition theorem type for certain "exponential–trigonometric type maps" and in particular an interesting recomposition tool for radioactive and pseudo–radioactive processes in the chemical setting.

The statement of our main result is the following.

Theorem 1. *Let $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be a bounded sequence of positive real numbers such that*

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{\log\lambda_k}{k}\right| < \infty.$$

Let $\sigma_\lambda(t) := \prod_{k\in\mathbb{Z}} \lambda_k^{\mathrm{sinc}\,(t-k)}$. Then, σ_λ is an analytic function and for every $t\in\mathbb{R}$ the following equality holds

$$\sigma_\lambda(t) = \lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}}\,\mathrm{sinc}\,(t-k)\right)^n.$$

Thus, the function σ_λ holds property $\mathcal{P}$ for $\tau = 1$.

As a consequence of our theorem we obtain the results of the paper [4].

Corollary 2. *The following maps hold property $\mathcal{P}$ for $\tau = 1$:*

(1) $f_1(t) = e^{\mathrm{sinc}\,(x-t)}$ *for every given $x\in\mathbb{R}\setminus\mathbb{Z}$,*
(2) $f_2(t) = e^{\frac{\cos(\pi t)}{t}-\frac{\sin(\pi t)}{\pi t^2}}$,
(3) $f_3(t) = e^{\frac{\cos(\pi t)}{1+t^2}+\frac{\cosh\pi}{\sinh\pi}\frac{t\sin(\pi t)}{1+t^2}}$.

The paper is divided into four sections. Section 2 is devoted to the study of the convergence and analyticity of σ_λ. In section 3 we shall prove Theorem 1 and finally in section 4 we shall present as application the proof of Corollary 2. We underline that there are many type of maps which belong to the σ_λ morphology and they are potential candidates of verifying Theorem 1 which is a huge source of applications.

2. Convergency and analyticity of σ_λ

Let $\lambda : \mathbb{Z} \longrightarrow \mathbb{R}^+$ be a sequence of positive real numbers making convergent, in the Chauchy principal value sense, the infinite product

$$\prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)}, \tag{2}$$

where we are using the following definition of convergency (see for instance [20, page 634]).

Definition 3. *An infinite product* $\prod_{n\in\mathbb{Z}} p_n$ *with* $p_n \neq 0$ *is called convergent if there exists* $\lim_{n\to\infty} \prod_{i=-n}^{n} p_i = \lim_{n\to\infty} P_n = P \neq 0$. *Moreover,*

a) *if* $P = 0$ *and* $p_i \neq 0$ $\forall i \in \mathbb{Z}$ *the product is called divergent to* 0.
b) *if* $P = 0$ *and* $p_i = 0$ *for some* i *the product is called convergent to* 0.

Therefore, since in our case $\lambda_k \neq 0$ for every $k \in \mathbb{Z}$, when we say that the product (2) is convergent we assume that this product is different from 0.

The function generated by the sequence λ through the previously quoted product will be called σ_λ

$$\sigma_\lambda(t) = \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)}, \tag{3}$$

and we denote by L_λ its logarithm, i.e.

$$L_\lambda(t) = \sum_{k\in\mathbb{Z}} \log\lambda_k \operatorname{sinc}(t-k) = \frac{\sin(\pi t)}{\pi} \sum_{k\in\mathbb{Z}} \frac{(-1)^k \log\lambda_k}{t-k}. \tag{4}$$

Note that since $\prod_{k\in\mathbb{Z}} {\lambda_k}^{\operatorname{sinc}(t-k)}$ is different from zero at every point the study of the product which defines σ_λ is equivalent to the study of the cardinal series

$$\sum_{k\in\mathbb{Z}} \log\lambda_k \operatorname{sinc}(t-k) \tag{5}$$

(see [16, page 380]).

Let denote by Λ the set of all sequences of positive real numbers for which the product (2) is convergent, i.e.,

$$\Lambda = \left\{ \lambda : \mathbb{Z} \longrightarrow \mathbb{R}^+ ; \sigma_\lambda(t) = \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)} \in (0,\infty), \quad \forall t \in \mathbb{R} \right\},$$

or equivalently

$$\Lambda = \left\{ \lambda : \mathbb{Z} \longrightarrow \mathbb{R}^+ ; \sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k) \in \mathbb{R}, \quad \forall t \in \mathbb{R} \right\}.$$

For our purpose of studying the existence and analyticity of the function σ_λ, and using the notation given by (4), we shall study in fact conditions which guarantee the existence and analyticity of the function L_λ.

The prove of the analyticity of σ_λ is based in a very well–known result on cardinal series, see for instance [14] or [24, page 22].

Theorem 4. *If a cardinal series converges for a non–integer value of $t \in \mathbb{C}$, then it converges uniformly on every compact set of the t–complex plane to an entire function on t.*

Therefore, every convergent cardinal series generates an analytic function and we have as an immediate consequence of Theorem 4 the following result.

Corollary 5. *If $\lambda \in \Lambda$, then the function σ_λ given by (3) is analytic. Conversely, if there exists $t_0 \notin \mathbb{Z}$ such that $\prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t_0-k)}$ is convergent then $\lambda \in \Lambda$.*

Now, we endeavor to present some results which allow us to verify in the practice and in a relatively easy way the convergence of the series (5).

Proposition 6. *Let λ be a bounded and symmetric sequence of positive real numbers. Then, the series $\sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k)$ converges for every $t \in \mathbb{R}$, i.e., $\lambda \in \Lambda$.*

Proof. Since the sequence $\{\log \lambda_k\}_{k\in\mathbb{Z}}$ is bounded, the character of the cardinal series coincides with the character of the series

$$\sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{\log \lambda_k - \log \lambda_{-k}}{k}, \tag{6}$$

(see [23]).

The proof follows from the convergency of (6) which comes from the symmetry of the sequence $\{\log \lambda_k\}_{k\in\mathbb{Z}}$. □

Remark 7. *We underline that Proposition 6 does not work without the symmetry hypothesis. Indeed, let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be the sequence defined by*

$$\lambda_k = \begin{cases} e & \text{if } k \geq 1 \text{ odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Obviously this is a non–symmetric bounded sequence and the cardinal series is not convergent because

$$\sum_{k\in\mathbb{N}} (\log \lambda_k \operatorname{sinc}(t-k) + \log \lambda_{-k} \operatorname{sinc}(t+k)) = \sum_{k\in\mathbb{N}} \log \lambda_k \operatorname{sinc}(t-k)$$
$$= \frac{\sin(\pi t)}{\pi} \sum_{k\in\mathbb{N}} \frac{(-1)^k \log \lambda_k}{\pi(t-k)} = \frac{\sin(\pi t)}{\pi} \sum_{k\in\mathbb{N}} \left(\frac{\log \lambda_{2k}}{t-2k} - \frac{\log \lambda_{2k-1}}{t-(2k-1)} \right)$$
$$= \frac{\sin(\pi t)}{\pi} \sum_{k\in\mathbb{N}} \frac{1}{(2k-1)-t}.$$

Proposition 8. *Let λ be a symmetric sequence of positive real numbers. If exists $a < 1$ such that $\left\{ \frac{|\log \lambda_k|}{k^a} \right\}_{k\in\mathbb{N}}$ is bounded, then the series $\sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k)$ converges for every $t \in \mathbb{R}$, i.e., $\lambda \in \Lambda$.*

Proof. Using the definition of the cardinal sinus function and the fact that λ is a symmetric sequence we have

$$\sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k) = \lambda_0 \operatorname{sinc}(t) + \frac{2t\sin(\pi t)}{\pi} \sum_{k\in\mathbb{N}} (-1)^{k+1} \frac{\log \lambda_k}{k^2 - t^2}.$$

By hypothesis there exist $a < 1$ and $C > 0$ such that

$$\sum_{k\in\mathbb{N}} \left| (-1)^{k+1} \frac{\log \lambda_k}{k^2 - t^2} \right| = \sum_{k\in\mathbb{N}} \left| \frac{\log \lambda_k}{k^2 - t^2} \right| \leq C \sum_{k\in\mathbb{N}} \left| \frac{k^a}{k^2 - t^2} \right| < \infty.$$

And thus, $\lambda \in \Lambda$. □

The following result is an adaptation of known results for cardinal series and could be useful in the practical sense. To prove Proposition 9 part 1) see [10]. For part 2) see [11].

Proposition 9. *The following statements hold:*

1) *If* $\sum_{k\in\mathbb{N}}(-1)^k \log\lambda_k$ *and* $\sum_{k\in\mathbb{N}}(-1)^k \log\lambda_{-k}$ *are convergent or finitely oscillate, then the series* $\sum_{k\in\mathbb{Z}} \log\lambda_k \operatorname{sinc}(t-k)$ *represents an analytic function.*
2) *If* $\sum_{k\in\mathbb{N}} \frac{(-1)^k \log\lambda_k}{k}$ *and* $\sum_{k\in\mathbb{N}} \frac{(-1)^k \log\lambda_{-k}}{k}$ *are convergent, then the series* $\sum_{k\in\mathbb{Z}} \log\lambda_k \operatorname{sinc}(z-k)$ *uniformly converges on compact sets to an entire function.*

Our main results works for bounded sequences $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ of positive real numbers satisfying

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{\log\lambda_k}{k}\right| < \infty. \tag{7}$$

Let $\mathcal{L}$ be the following set:

$$\mathcal{L} = \left\{ \lambda : \mathbb{Z} \longrightarrow \mathbb{R}^+ ; \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{\log\lambda_k}{k}\right| < \infty \right\},$$

In the next results we will try to link the absolute convergency of the series under our study and to belong to the set $\mathcal{L}$.

Proposition 10. *The series* $\sum_{k\in\mathbb{Z}} \log\lambda_k \operatorname{sinc}(t-k)$ *is absolutely convergent for every* $t\in\mathbb{R}$ *if and only if the sequence* $\lambda\in\mathcal{L}$.

Proof. Since

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\log\lambda_k \operatorname{sinc}(t-k)| &= \sum_{k\in\mathbb{Z}} \left|\frac{\log\lambda_k \,\sin(\pi(t-k))}{\pi(t-k)}\right| \\ &\leq \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \left|\frac{\log\lambda_k}{t-k}\right|\end{aligned}$$

the condition (7) is sufficient. The converse is obtained directly taking $t = \frac{1}{2}$. ☐

Corollary 11. $\mathcal{L}$ *is a proper subset of* Λ *(*$\mathcal{L}\subset\Lambda$*).*

Proof. The inclusion $\mathcal{L} \subseteq \Lambda$ is a direct consequence of Proposition 10. To prove that the inclusion is proper we consider for instance the constant sequence different from one $\lambda = \{c\}_{k\in\mathbb{Z}}$.
Indeed, $\lambda \in \Lambda$ because

$$\sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k) = \log c \sum_{k\in\mathbb{Z}} \operatorname{sinc}(t-k) = \log c$$

and therefore

$$\sigma_\lambda(t) = \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)} = c.$$

But $\lambda \notin \mathcal{L}$ since the series

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| = \log c \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|\frac{1}{k}\right|$$

is not convergent. □

Remark 12. *Other stronger conditions which guarantee too the absolute convergency of the series (4), are for instance*

a) *there exists* $p > 1$ *tal que* $\sum_{k\in\mathbb{Z}} |\log \lambda_k|^p < \infty$

b) $\sum_{k=2}^{\infty} (|\log \lambda_k| + |\log \lambda_{-k}|) \dfrac{\log k}{k} < \infty,$

see [11], [18] *or* [24, page 21] *respectively.*

3. Proof of Theorem 1

In this section we shall prove Theorem 1. We start with an auxiliary result.

Lemma 13. *Let* $\{\lambda_k\}_{k\in\mathbb{Z}}$ *be a sequence of positive real numbers. For every* $k \in \mathbb{Z}$ *and for every* $n \in \mathbb{N}$ *then*

$$\left|\lambda_k^{\frac{1}{n}} - 1\right| \leq \frac{|\log \lambda_k|}{n} \max\left\{1, \lambda_k^{\frac{1}{n}}\right\}.$$

In particular, if $\{\lambda_k\}_{k\in\mathbb{Z}}$ *is bounded above then there exists* $L \geq 1$ *such that*

$$\left|\lambda_k^{\frac{1}{n}} - 1\right| \leq L \frac{|\log \lambda_k|}{n}.$$

Proof. Applying the Mean Value Theorem to the function $f(x) = \lambda_k^x$ in the interval $\left[0, \frac{1}{n}\right]$ follows that there exists $c \in \left(0, \frac{1}{n}\right)$ such that

$$\left|\lambda_k^{\frac{1}{n}} - 1\right| = \frac{|\log \lambda_k|}{n} \lambda_k^c.$$

Given that if $0 < \lambda_k \leq 1$, then $\lambda_k^c \leq 1$. Finally, if $\lambda_k > 1$ then $\lambda_k^c < \lambda_k^{\frac{1}{n}} < \lambda_k$ ending the proof. □

<u>*Proof of Theorem 1.*</u> The analyticity follows from Theorem 4. For simplicity we shall use the following notation:

$$h(t, n) := \sum_{k \in \mathbb{Z}} \lambda_k^{\frac{1}{n}} \operatorname{sinc}(t - k).$$

In this setting, what we have to prove is

$$\lim_{n \to \infty} (h(t, n))^n = \prod_{k \in \mathbb{Z}} {\lambda_k}^{\operatorname{sinc}(t-k)}.$$

Indeed, note that from Proposition 10, the condition required (7) make absolutely convergent the series $\sum_{k \in \mathbb{Z}} \log \lambda_k \operatorname{sinc}(t - k)$. Moreover, since λ is bounded, by Lemma 13, there exists $L \geq 1$ such that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t - k) \right| &\leq L \sum_{k \in \mathbb{Z}} \left| \frac{\log \lambda_k}{n} \right| |\operatorname{sinc}(t - k)| \\ &\leq L \sum_{k \in \mathbb{Z}} |\log \lambda_k \operatorname{sinc}(t - k)| < \infty. \end{aligned}$$

Therefore, using on the one hand the Dominated Convergent Theorem and on the other hand the fact $\lim_{n \to \infty} \left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t - k) = 0$, we have that

$$\begin{aligned} \lim_{n \to \infty} (h(t, n) - 1) &= \lim_{n \to \infty} \left(\sum_{k \in \mathbb{Z}} \lambda_k^{\frac{1}{n}} \operatorname{sinc}(t - k) - \sum_{k \in \mathbb{Z}} \operatorname{sinc}(t - k) \right) \\ &= \lim_{n \to \infty} \sum_{k \in \mathbb{Z}} \left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t - k) \\ &= \sum_{k \in \mathbb{Z}} \lim_{n \to \infty} \left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t - k) = 0, \end{aligned}$$

from which we obtain

$$\lim_{n \to \infty} h(t, n) = 1. \tag{8}$$

Now, proceeding in an analogous way and using again Lemma 13, the condition (7) guarantees

$$\sum_{k\in\mathbb{Z}} \left| n\left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t-k) \right| \quad \leq \quad L \sum_{k\in\mathbb{Z}} |\log \lambda_k| |\operatorname{sinc}(t-k)| < \infty.$$

Thus, since $\lim_{n\to\infty} n\left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t-k) = \log \lambda_k \operatorname{sinc}(t-k)$, by the Dominated Convergent Theorem again we have

$$\begin{aligned} \lim_{n\to\infty} n(h(t,n)-1) &= \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} n\left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t-k) \right) \\ &= \sum_{k\in\mathbb{Z}} \lim_{n\to\infty} n\left(\lambda_k^{\frac{1}{n}} - 1\right) \operatorname{sinc}(t-k) \\ &= \sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k). \end{aligned}$$

In consequence, from the previous expression joint with (8), we conclude

$$\lim_{n\to\infty} (h(t,n))^n = e^{\sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k)} = \prod_{k\in\mathbb{Z}} {\lambda_k}^{\operatorname{sinc}(t-k)},$$

ending the proof. □

Remark 14. *The hypothesis of Theorem 1, λ bounded and $\lambda \in \mathcal{L}$, is a sufficient condition but not necessary. Indeed, as a counterexample we can consider for instance any constant sequence $\lambda = \{c\}_{k\in\mathbb{Z}}$ with $c \neq 1$. Following the proof of Corollary 11, $\lambda \notin \mathcal{L}$ and $\sigma_\lambda(t) = c$ satisfies property $\mathcal{P}$ by* [2]. *Other counterexample less trivial is the one stated in* [9]. *It is proved that the function $e^{\cos(\pi t)}$ satisfies property $\mathcal{P}$. Now, note that $e^{\cos(\pi t)}$ can be expressed as σ_λ for $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ given by $\lambda_k = e^{(-1)^k}$ and $\lambda \notin \mathcal{L}$. Indeed, $\lambda \notin \mathcal{L}$ because the series $\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| = \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \left|\frac{1}{k}\right|$ is divergent.*

4. Applications of Theorem 1

The aim of this section is to prove, as a application of Theorem 1, Corollary 2 proving that such maps can be generated as σ_λ ones for a properly chosen sequence $\lambda \in \mathcal{L}$.

Proof of Corollary 2. **(1)** Let $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}} = \left\{e^{\operatorname{sinc}(x-k)}\right\}_{k\in\mathbb{Z}}$ for every given $x \in \mathbb{R} \setminus \mathbb{Z}$.

Clearly λ is a bounded sequence and $\lambda \in \mathcal{L}$ because

$$\begin{aligned}\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left|\frac{\log\lambda_k}{k}\right| &= \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left|\frac{\operatorname{sinc}(x-k)}{k}\right| = \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left|\frac{\sin(\pi(x-k))}{\pi(x-k)k}\right| \\ &\leq \frac{1}{\pi}\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left|\frac{1}{k(x-k)}\right| < \infty.\end{aligned}$$

Using that $\operatorname{sinc}(x-t) = \sum_{k\in\mathbb{Z}} \operatorname{sinc}(x-k)\operatorname{sinc}(t-k)$ (see [4]), we have

$$\sum_{k\in\mathbb{Z}}\log\lambda_k \operatorname{sinc}(t-k) = \sum_{k\in\mathbb{Z}}\operatorname{sinc}(x-k)\operatorname{sinc}(t-k) = \operatorname{sinc}(x-t)$$

and it follows that

$$\sigma_\lambda(t) = \prod_{k\in\mathbb{Z}}\lambda_k^{\operatorname{sinc}(t-k)} = e^{\operatorname{sinc}(x-t)}.$$

Therefore, by Theorem 1 we obtain that $f_1(t) = e^{\operatorname{sinc}(x-t)}$ is an analytic function and satisfies the property $\mathcal{P}$ for $\tau = 1$.

(2) Let $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ defined by

$$\lambda_k = \begin{cases} 1 & \text{if } k = 0, \\ e^{\frac{(-1)^k}{k}} & \text{si } k \neq 0. \end{cases}$$

Obviously $\{\lambda_k\}_{k\in\mathbb{Z}}$ is bounded and

$$\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left|\frac{\log\lambda_k}{k}\right| = 0 + \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left|\frac{(-1)^k}{k^2}\right| < \infty.$$

Since $\sum_{k\in\mathbb{Z}}\log\lambda_k \operatorname{sinc}(t-k) = \frac{\cos(\pi t)}{t} - \frac{\sin(\pi t)}{\pi t^2}$ (see [22]), we have

$$\sigma_\lambda(t) = \prod_{k\in\mathbb{Z}}\lambda_k^{\operatorname{sinc}(t-k)} = e^{\frac{\cos(\pi t)}{t} - \frac{\sin(\pi t)}{\pi t^2}}.$$

So, by Theorem 1 the map $f_2(t) = e^{\frac{\cos(\pi t)}{t} - \frac{\sin(\pi t)}{\pi t^2}}$ is an analytic function and satisfies the property $\mathcal{P}$ for $\tau = 1$.

(3) Let be the bounded sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}} = \left\{e^{\frac{(-1)^k}{1+k^2}}\right\}_{k\in\mathbb{Z}}$. Then $\lambda \in \mathcal{L}$ because

$$\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left|\frac{\log\lambda_k}{k}\right| = \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\left|\frac{1}{k(1+k^2)}\right| < \infty.$$

By [6] it is followed that

$$\sum_{k\in\mathbb{Z}} \log \lambda_k \operatorname{sinc}(t-k) = \frac{\cos(\pi t)}{1+t^2} + \frac{\cosh\pi}{\sinh\pi}\,\frac{t\sin(\pi t)}{1+t^2},$$

hence,

$$\sigma_\lambda(t) = \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)} = e^{\frac{\cos(\pi t)}{1+t^2} + \frac{\cosh\pi}{\sinh\pi}\,\frac{t\sin(\pi t)}{1+t^2}} = f_3(t).$$

Thus, by Theorem 1 the proof is over. □

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